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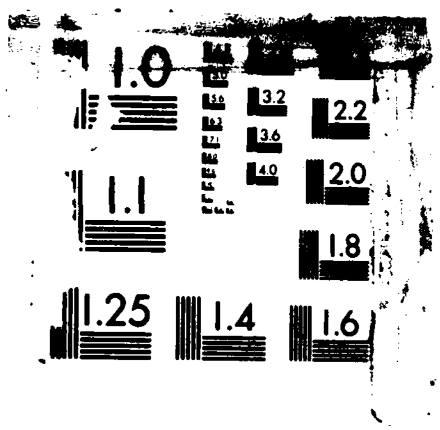
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A PARTIAL DIFFERENTIAL EQUATION MODEL DESCRIBING THE THREAT TO AN AIRCRAFT IN A ONE-ON-MANY SCENARIO

THESIS

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THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

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Henry G. Birkdale, B.S.

Second Lieutenant, USAF

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Abstract

This research investigates the feasibility of describing the threat potential at a point in a specified air defense zone as a function of position and time while stipulating that the function is a solution to a partial differential equation. The mission, the aircraft, and the hostility of the air defense zone are incorporated into the forcing function, as well as the initial and boundary conditions which are needed to solve the partial differential equation. A constant speed and altitude of the aircraft is assumed.

In order to validate the partial differential equation used to generate the threat data a scheme for determining the path of least threat through the threat data is proposed. The calculus of variations, a branch of mathematics concerned with the optimization of an integral, is used to find the curve $x(t)$ and $y(t)$ which minimizes the total threat of the flight path. Preliminary results indicate that it does appear feasible to describe the threat as above but further validation is required.

I. Introduction

Issue

Currently, military analysts use computer simulation to analyze the attrition of friendly aircraft penetrating an enemy air defense zone. Although simulation offers valuable insights into the nature of this particular form of aerial combat, it has shortcomings. First, the flight path of the penetrating aircraft is usually one of many inputs to the simulation model. What if the objective was to determine the optimal flight path through the air defense zone in the sense of the path of least threat? In order to answer this question the analyst would have to investigate a large number (technically infinite) of flight paths, clearly a drawback.

Another relative limitation in trying to determine the optimal flight path is the measure of effectiveness used to describe the level of threat faced by the aircraft. In most cases this threat level is either the probability of individual aircraft kill or the attrition rate, which is the number of aircraft lost per total number of attacking aircraft and which obviously depends on the probability of kill. This probability of kill is usually calculated for the flight path or a segment of the path, an individual surface-to-air missile (SAM), or a single round from an anti-aircraft artillery (AAA) gun. In calculating the probability of kill one must specify other probabilities to include the

probability of detection, launch, fuzing, hit and other variables. With this amount of data it would not be feasible to specify the probability of kill at every point in the air defense zone, for every velocity of the aircraft, at all times during the flight.

What is needed then is a way to describe mathematically the threat level that an aircraft faces while operating in an air defense zone. If this were possible then the problem of determining the path of least threat could be formulated as a calculus of variations problem which is concerned with finding the curve which optimizes an integral. In this case the curve would be the flight path and the integral would be the integral of threat level over time. The output would be the path of "least threat".

What is meant by "threat"? A surface-to-air missile is certainly a threat to an aircraft. For that matter so is an early warning radar but in a less lethal way. For the purposes of this thesis a source of threat is not the same as the "threat". A SAM, a AAA, ground controlled interceptors are all sources of threat. These physical objects provide the feeling of threat that one faces when operating in a combat environment.

> There is nothing, absolutely nothing, to describe what goes on inside a pilot's gut when he sees a SAM get airborne.

Commander Randy "Duke" Cunningham as quoted in (Shaw, 1985:58)

What is the threat a function of? Certainly a knowledge

of the threat source environment is important. Knowing what types, how many, the effectiveness, and where the threat sources are is critical. In addition the threat level changes as a function of the pilot and his mission. An A-10 pilot at 1000 feet has an entirely different opinion of an SA-7 than does a B-52 pilot at 30,000 feet.

The threat to an aircraft changes with time and with the state of the enemy air defenses. Consider an aircraft versus SAM encounter under two different scenarios. The first scenario consists of the SAM site at full battle stations with the aircraft having no element of surprise. Under the second scenario the aircraft surprises the SAM site. The state of the defense is such that the threat provided by the SAM site under the second scenario is less than the threat provided against the first. The state of the defense also changes with time and with the state of the penetrating aircraft. To see this again consider an aircraft penetrating an air defense zone that consists of one SAM site. At first the aircraft is far enough away that the probability of being shot down is almost zero. As the aircraft approaches the zone the site begins tracking the aircraft and the threat is unquestionably increasing. If the aircraft does not alter course, then the threat will continue increasing. Assume now that the aircraft commander realizes that he is being tracked by the site. He takes evasive action by climbing in altitude until he is out of reach of the missile. Once the defense

commander is aware that the aircraft is out of reach, he might still track the aircraft but he will not shoot at it. This would be a waste of a missile. The state of the enemy defense changed. This change was due in part to the state(altitude) of the aircraft changing. Most importantly, the threat to the aircraft was constantly changing.

An ordinary differential equation is an equation describing the relationship between an independent variable (say x), a dependent function y(x), and the derivatives of y with respect to x. (Kaplan, 1984:533) When y is a function of more than one independent variable then a partial differential equation expresses a relationship between y, the dependent variable, and the partial derivatives of y with respect to the independent variables. (Kaplan, 1984:635)

Problem Statement

Can the threat to an aircraft operating in an air defense zone be expressed as a mathematical equation with the stipulation that this equation is the solution to a partial differential equation? The use of partial differential equations in modeling systems has been used quite extensively with the greatest success occurring in engineering and the physical sciences.

In order to obtain a solution to certain partial differential equations the initial and boundary conditions as well as forcing, or environmental, functions must be specified. Can the state of the defenses as well as certain

offensive tactics (jamming for instance) be described by the initial and boundary conditions and forcing functions? Finally, can the pilot's opinion of the lethality of the threat sources with respect to his aircraft and mission be translated into the initial conditions required to solve the partial differential equation?

Discussion of the Literature.

"A literature review gives readers the background and justification for the current study, and it helps the reader understand how the current research will contribute to further knowledge of the subject" (Skelton, 1986:1). This literature review is divided into two main topics. The first topic to be addressed will be mathematical modeling with an emphasis on the use of partial differential equations. The second area to be discussed will be the current modeling of aircraft versus air defense systems.

Mathematical Review. According to Webster mathematics is "...the group of sciences ... dealing with the quantities, magnitudes, and forms, and their relationships, attributes,etc., by the use of numbers and symbols". Mathematics can also be thought of as a language; a very efficient language for modeling purposes. As an example consider Maxwell's equations. These are four relatively simple equations in electromagnetic theory that describe the relationship between electric and magnetic fields. To describe this relationship in English requires hundreds of

pages of text. This is just one example of the efficiency of mathematics; science and engineering are filled with many more. In order to converse in a language however requires an understanding of the terms and concepts of that language. Some of the mathematical terms and concepts used in this thesis appear very complicated. The reason for this is that the real world, and in particular the specific system to be modeled, is complicated. Although it is impossible to define every term an understanding of the basic terms should facilitate an understanding of the material.

"The key to a useful solution of any complex problem is the skillful choice of the simplest possible model which preserves the salient system properties under investigation" (Shinar, 1980:9). There are times however when this sought after goal of the simplest possible model is not realized. The reason for this is that complex problems arise due to the fact that the problem is part of a complex system. Difficulties with complex systems arise due to the interaction of different purposes and goals (Selfridge et al, 1984:1-4) or because the boundaries of the system are not clear. Either way the key to solving the problem is an understanding of the relationship between the system and the problem (Dorny, 1975:11-12). At times the complexity of the system is so great that the solution of a problem appears intractable. Why model? The answer is simply because the need is there.

Kubrusley offers the following method for identifying the mathematical system which best models the real system. Let S be the real system to be identified and let

$$F = \{C_i, \text{ i is an element of } I\}$$

where

C_i is a class of models,

F is a set of classes of models and,

I is an index set. (Kubrusley, 1977:509-510)

The system identification can be accomplished in two steps. The first step is the system characterization which is the determination of a set, F, of a class of models. This step is concerned with the different forms of the description of a system. For example a system can be described by a differential equation, a partial differential equation, a state equation, or a simple algebraic equation. (This is not an exhaustive list.) The characterization step is also concerned with the form of the description of the system. For instance, can we accurately describe the system at discrete intervals of time or must we continuously describe the system? Can the state of the system take on only a finite number of values or an infinite number of values?

The second step in system identification is called the classification step. Once we have chosen a set F of a class of models we must chose an element, C_i, of F. Assume that we have chosen F to be the set of all partial differential equations. Then C_i might be the set of all nonlinear partial differential equations, or it might be the set of all linear

partial differential equations with constant coefficients. (Again, this list is not exhaustive.) Finally we must choose an element of C_i. "Given a class C_i (with each member of C_i completely characterized) the problem is to determine a model M in C_i which is 'equivalent' to the system S" (Kubrusley, 1977:513). The method of determining equivalence is usually the minimization of the difference between what is actually observed from the system and what is predicted from the model.

The set of classes of models that was chosen to describe the threat to an aircraft was the set of partial differential equations (PDE). Although the threat to an aircraft could not be described in closed form for every contingency it was hypothesized that the threat was a solution to a particular partial differential equation. In this way different contingencies could be modeled by the different boundary and initial conditions needed to solve the PDE.

A review of the literature indicates that the threat to an aircraft has not been modeled as a solution to a PDE. The reason for choosing this methodology to describe the threat is that differential equation models, although not always simple and convenient, are more descriptive than, for example, algebraic equations. "Differential equations permit a more detailed study of the dynamic process of the formation of the quantity" , in this case the threat, "being forecast..." (Chuyev and Mikhaylov, 1975:101).

There is one main problem with a partial differential equation and that is in the determination of a solution. The determination of a solution, especially an analytic solution, can be an arduous task except for the simplest of cases. Depending on the form of the equation a closed form solution may not be available in which case numerical methods must be used. Therefore, once an equation that is equivalent to the system S has been identified in many cases it is desirable to simplify the equation. In some cases it is possible to let the spatial variables take on only a discrete set of values. Assuming the system is time dependent this results in a set of ordinary differential equations which are usually easier to solve. When both time and space derivatives in a PDE are approximated by finite differences the result is a set of difference equations (Robinson, 1971:374).

The mathematical literature on solution techniques for partial differential equations and on the use of partial differential equations in modeling is quite extensive. The initial desire of the author was to obtain closed form expressions for the solutions of a partial differential equation. Because of this desire the literature review centered on solution techniques.

A general second order partial differential equation is an equation that contains two derivatives in at least one of the variables and has the form

$$AU_{xx} + 2BU_{xy} + CU_{yy} + DU_{x} + EU_{y} + FU + G = 0$$

where U_x is the partial derivative of U with respect to x. The coefficients A, B, C, etc., can be constant or can also be functions of the independent variables.

We classify partial differential equations by the value that $d = AC - B^2$ takes on.

$d < 0$	elliptic
$d = 0$	parabolic
$d > 0$	hyperbolic

A system that can be described by a partial differential equation is called a distributed parameter system (DPS). A system that can be modeled by an ordinary differential equation is called a lumped parameter system (LPS).

For a good introduction to solution techniques see (Farlow, 1982) although all of the examples and problems are discussed using only one spatial dimension. There is little theory and the text is arranged in forty-seven short (a few pages) lessons. For those with a course in ordinary differential equations but with no background in partial differential equations this is a good text to start.

Advanced texts on solution techniques for partial differential equations are (Bateman, 1944), (Duff and Naylor, 1966), and (Courant and Hilbert, 1953). These references do not require a background in physics although it would certainly help. Both Bateman and Duff solve higher dimensional problems with Bateman's text the more detailed due to his inclusion of the solutions of three dimensional

problems as well as partial differential equations in five different coordinate systems. Bateman also includes numerous examples from physics and engineering. Courant and Hilbert's text is the most advanced of the three with existence and uniqueness theorems given.

According to Gustafson "...there has been a need for a satisfactory introduction to partial differential equations and Hilbert space methods at the undergraduate level (Gustafson, 1980:vii)". Accepting this the text appeared to be suited more towards graduate students in mathematics than undergraduates. In fact one of the stated subobjectives was to introduce graduate students to the subject. Although the book was interesting it was not suited as a reference text for solution methods.

This concludes the discussion of references in the literature concerning solution techniques for partial differential equations. In no way was the above list complete. In fact many helpful methods were found in purely physics books. Most advanced books on heat and mass transfer will have a variety of solution techniques for parabolic differential equations. Solution techniques for elliptic and hyperbolic equations can be found in most texts on electricity and magnetism. This also concludes the presentation of the methodology used to formulate a complex system in terms of a mathematical model. The remainder of this literature search will consist of a survey of air defense models.

Review of Air Defense Models. There are three different types of SAM models: fire analyzers, engagement simulators, and engineering models. The literature on fire analyzers and engagement simulators was reviewed. "Fire analyzers are models which estimate the spatial volume within which an aircraft is susceptible to fire by an air defense unit in terms of engagement parameters. Engagement parameters are variables such as aircraft velocity, aircraft altitude, intercept distance, missile launch angle, firing delay, etc" (Reid, 1982:128). For the most part these models have an aircraft trajectory that is straight and level. The primary usage of the fire analyzers are to assess the aircraft vulnerability and to assess the number of launches achievable by a SAM site. They usually do not address ECM (Reid, 1982:129).

The engagement simulators estimate the engagement miss distances and the probability of kill. These models take into account flight parameters, design variables, and launch conditions. They offer an advantage over fire analyzers in that the engagement simulators "...do not require as many engineering estimates or standard value assumptions; the simulation generates its own values based on previous calculations in the simulation" (Reid, 1982:129).

The missile versus aircraft encounter can be thought of as a microscopic model when considered in the context of an aircraft versus air defense unit scenario. The pursuit and evasion process can be divided into three phases. First is

the initial acquisition phase where the target is being tracked. Next is the main pursuit phase which consists of the missile on its way to the aircraft. The final phase is called, appropriately enough, the end-game. (Shinar, 1980:3) Shinar's initial attempt at modeling the missile versus aircraft encounter was limited to two dimensional motion. Even with this simplification the numerical solutions were tedious, time consuming, and misleading. (Shinar, 1980:2) The pursuit evasion problem can be approached in two ways. It can be looked at as a zero sum differential game or it can be approached as two distinct control problems. (Shinar and others, 1979:353) Either way the problem is very complex. The reasons for this complexity are the three dimensional nature of the battle and the fact that the missile is guided according to complicated control laws. Some of the simplifications that must be made to allow for reasonable computation time are that both the missile and the aircraft fly at constant speed, that initially the aircraft is nonmanuevering, and that all trajectories can eventually be linearized.

Counter air operations have been simulated on a computer and they are most helpful in that they can help identify the important variables and what type of tactics should be employed against a certain threat. "Threats can also dictate that one tactic be favored over another. Typical threat systems defending an airfield incorporate some combination of

antiaircraft artillery and surface-to-air missile" (Foley and Gress, 1984:14).

The number of models simulating aircraft versus SAMs or aircraft versus antiaircraft artillery is quite extensive. Examples of one-on-one models include GUN, an antiaircraft artillery model which simulates a penetrator flying a preprogrammed flight path flying against a AAA, P001, which also simulates the AAA encounter, and ATSAM, the Advanced Threat Survivability Analysis Model. The measure of effectiveness for these models vary. GUN uses the accumulation of the number of rounds that come within the lethal radius of the target. P001 uses the probability of single shot kill, while ATSAM measures survivability as a function of penetrator parameters (including ECM), altitude, and weapons systems parameters. (AFEWC, 1985:ch 2)

Some of the one-on-many models include OZ, or Optical Zinger, which uses the outputs from P001 and TAC Zinger, another SAM model (AFEWC,1985: ch2 185). GREAT SAM, BIG SAM, and BOLD SAM are models of different SAM system (Battilega and Grange, 1984:136). "The Avionics Air Defense Evaluation Model (AADEM) is a one-sided, deterministic (with probabilistic end game) time/event based modular campaign model which simulates the penetration of aircraft in a scenario of tactical or strategic jamming aircraft, hostile interceptors, ground based threats, plus C^2 networks" (AFEWC, 1985: ch2 51). Again the outputs from these models are

usually survivability as a function of sortie parameters, threat laydown and characteristics.

There are of course other models but the main relationship that the one-on-one and one-on-many models share is the measure of effectiveness. Usually the measure is the missile miss distance or the probability of kill or some function of the probability of kill. As a final example there is the Force Level Automated Planning System (FLAPS). "Threat effectiveness is specified as the negative log of the probability of survival per second at each up-range/ down-range and cross range position within the threat's radius" (SCT, 1986:ch III 20). Although these are of course related to the threat level that an aircraft faces while flying through an air defense zone there are two main problems with these measures. First, they are not continuous functions of position and time. If the objective was to determine the optimal flight path through the air defense zone then ideally one would need to know the threat at every point (x,y) for all t. Second, there is no effective way to add together the threat levels due to multiple threat sources.

This concludes the literature survey. The work done in this thesis is now described.

Assumptions

The main assumption of this thesis is that for an aircraft operating in an air defense zone there exists a

scalar valued function that is indicative of the threat faced by the aircraft and that this function is the solution to a partial differential equation. In general the threat will be a function of an infinite number of variables but it will be assumed that the threat level changes with aircraft position in the zone, velocity, and time. A further assumption is that the pilot, given a knowledge of the types and position of the threat sources will be able to rank order these threat sources in order of increasing lethality as well as assign numerical ratings for each threat source.

In order to test the validity of the function generating the threat level the calculus of variations will be used to determine the path of least threat through the zone as a function of position and time. This is certainly not a rigorous validation. If the flight path predicted is counter intuitive, then this would indicate that the partial differential equation used to generate the threat level is not the equation that most accurately simulates the real world. Therefore it will be assumed that any flight path predicted by the calculus of variations will be feasible. That is, there will be no constraints on aircraft performance or speed.

It will be assumed that all distances and time units can be scaled to range between zero and one. As a consequence of this no time or distance units will be exhibited.

Finally, although the partial differential equation will

allow for three dimensional motion, the analysis will assume a single aircraft operating at constant altitude.

Research Objective

It is the purpose of this research to investigate the feasibility of modeling the threat to an aircraft performing in an air defense zone as the solution to a partial differential equation. This differential equation will describe how the threat level changes with time and with the state of the penetrating aircraft.

In order to model the threat to an aircraft as a partial differential equation certain subobjectives must be met. These are:

1) To postulate basic principles which describe the concept of threat,

2) To derive a partial differential equation from these postulates,

3) To relate the differential equation to a scalar valued function which is a measure of threat,

4) To obtain computer resources needed to solve not only a partial differential equation but also the ordinary differential equations which result from the calculus of variations,

5) To validate the predicted flight paths with intuition.

The rest of the thesis is divided into four chapters. Chapter two contains background information and the definition of terms. In chapter three more terms are defined and the partial differential equation used to generate the threat data is presented. Optimal flight paths are presented in chapter four and recommendations for further study are included in chapter five.

II. Background

Examples of Partial Differential Equation Models

A system, for the purposes of this thesis, can be thought of as a set of objects which interact. In most circumstances as the complexity of the system of interest increases the degree of difficulty in modeling that system also increases. Many complex systems arise due to the interactions of simple systems (Selfridge et al, 1984:3). An example of what, at first glance, appears to be a simple system is the missile versus aircraft encounter. In the context of one aircraft operating in an air defense zone which consists of multiple threats the missile versus aircraft encounter could be thought of as a microscopic model. In (Shinar et al, 1979) differential game theory has been used to predict the optimal aircraft maneuver needed to maximize the missile miss distance in the missile versus aircraft encounter. Yet these air to air combat models have not been applied due to the fact that there are too many simplifications (ie- constant speed, aircraft modeled as a mass point, linearized trajectory) of the real world resulting in an inaccurate model. Even with these simplifications the numerical methods needed to generate an optimal aircraft maneuver are tedious and time consuming. The missile versus aircraft encounter is, in fact, a complex system, due to the control laws which govern the missile.

Since the missile versus aircraft system is part of a larger, more complex system (the one-on-many scenario), then one would expect even greater difficulty modeling this complex system.

A complex system which consists of a large number of simple, discrete systems can sometimes be formulated mathematically by assuming that this large number of discrete systems is in fact one continuous system. As an example consider the diffusion of heat in a substance. The flow of heat in an object is due to the interactions of the millions of molecules which make up the object. Just because one is unable to model the interactions of each and every one of these discrete systems (the molecules) does not imply that the temperature of the object cannot be predicted. The diffusion, or heat, equation predicts how the temperature of a object changes with respect to both position and time. One can therefore describe the system (the temperature of the body) at any time t and any position provided the initial temperature distribution, the temperature at the boundaries of the object, and any sources or sinks of heat during the time of observation, are specified. The heat equation is a partial differential equation which describes how the state of the system evolves over time.

The heat equation is but one partial differential equation used to explain physical phenomena. There are many other examples of PDE models in science and engineering. Maxwell's equations in electricity and magnetism are partial

differential equations which describe how a changing electric field interacts with a changing magnetic field. The position of a point in a vibrating string can be described by the wave equation. The solution of Schrödinger's equation in quantum mechanics yields the probability of finding a particle in a region of space. The classical differential equations of science are intimately related. The solution to Poisson's equation describes the temperature distribution in a substance at steady state with a heat source as well as the electric potential due to a charge distribution.

Although the mathematics needed to solve a partial differential equation can at times be somewhat difficult, the real problem lies in trying to determine a PDE which accurately describes the system of interest. This is called structural identification. There are three general approaches to structural identification. The experimental approach is one in which an analysis of the inputs and outputs of the system is undertaken with the hope that some relationship between inputs and outputs can be identified. The experimental approach is applied usually to systems which are thought to be described by ordinary differential equations. The transfer function approach describes how a system changes as a result of discrete changes in the variables which influence the system. Finally, the axiomatic approach is one in which the laws that govern the system are analyzed with the hope that the state evolution process can be derived from these laws (Kubrusley, 1977:512-513).

These approaches can and do overlap. In the diffusion of heat in a substance early experiments indicated that if there were no heat sources in a body then the temperature of that body would eventually approach a constant. Other experiments resulted in the knowledge that if there was a temperature difference between two points in a body at some time t then at a later time T this difference had diminished, indicating that heat flowed from a point of higher temperature to one of lower temperature and not the other way around. From these observations scientists hypothesized that the rate of heat transfer in a substance was proportional to the temperature gradient, the chemical elements which made up the body, and the density of the body. These experiments paved the way for the hypothesis of theoretical laws and from these laws the heat equation was derived.

There is no physical entity which "measures" the threat. Because of this lack of hard experimental data the system identification stage must be modified. In this thesis terms will be def. ·d and "laws" (for lack of a better term) will be proposed so that the threat to an aircraft operating in an air defense zone can be described by a partial differential equation. Once the threat is specified the optimal path through the threat will be determined (a quasi-experiment) and this path will be looked at intuitively to determine the applicability of the threat equation.

Optimization of an Integral

In what follows it will be necessary to minimize an expression which involves the integral of a function. The calculus of variations is a branch of mathematics concerned with the optimization of an integral. If one wishes to find the curve x(t) which minimizes the integral:

$$J = \int_a^b \xi(t,x,x')dt$$

with $x'=dx/dt$ then x(t) must satisfy an ordinary differential equation (called Euler's equation) of the form

$$\partial\xi/\partial x - d/dt(\partial\xi/\partial x') = 0 \qquad \text{(Eq 2-1)}$$

The solution of this ordinary differential equation yields a curve x(t) that minimizes J. As an example suppose the objective is to find the curve x(t) which minimizes the distance between two points. (Assume both x and t are dimensionless.) The arc length of a curve is

$$(ds)^2 = (dx)^2 + (dt)^2$$

Therefore the integral to be minimized is

$$J = \int_a^b (1 + x'^2)^{1/2}\, dt$$

with $\xi = (1 + x'^2)^{1/2}$

Referring to (Eq 2-1)

$$\partial\xi/\partial x = 0$$

$$\partial\xi/\partial x' = x'/(1 + x'^2)^{1/2}$$

and

$$-d/dt(\partial\xi/\partial x') = 0$$

which implies that with respect to t

$$x'/(1 + x'^2)^{1/2} = \text{constant}$$

or

$$x' = dx/dt = \text{constant}$$

which is the equation of a straight line. (Boas, 1983:387)

Goals of the Model

The primary objective of this thesis is to describe the threat to an aircraft in an air defense zone as a function of position, time, and aircraft speed as well as finding the flight path of least threat through an air defense zone which consists of a variety of air defenses. This second objective can be thought of as a partial validation of the equation used to describe the threat. It is a partial validation because if the optimal flight paths are intuitively appealing then that would at least not eliminate the use of that equation as a description of the threat.

The survivability of an aircraft depends on many variables. In general the survivability of the aircraft depends on the type of mission, the support from friendly forces, and the intensity and effectiveness of the hostile environment (Ball, year:61). Any model that is used in predicting an optimal flight path through an air defense zone should take into account these variables.

In this thesis these variables are classified into three areas: aircraft variables, enemy variables and time. The aircraft variables are the mission, position, velocity, and

support from friendly forces (jamming, for instance). The enemy variables are the type, number, and position of the enemy defenses as well as the state of readiness of the defenses.

To incorporate all of these variables into a single algebraic equation would be extremely difficult, if not impossible. If it were possible the likelihood of utilizing the equation in different situations would be very small. The model therefore must be flexible enough so that it will be useful in different threat scenarios yet it must also incorporate the needed variables. The main hypothesis of this thesis is that there exists a scalar threat function ξ which describes the threat to an aircraft performing in a threat environment as a function of position, velocity, and time. In addition, ξ is also the solution to a partial differential equation. The strength of a partial differential equation model is that in order to generate a solution to a PDE you must have:

1. Initial conditions- The state of the system when $t=0$.
2. Forcing terms- any changes to the system during the period of observation.
3. Boundary conditions- the state of the system at the boundaries during the period of observation.

If any one of the above conditions change then the solution to the PDE can and usually will change.

To put the above mathematical background in context with the objectives of this thesis is relatively straightforward. We assume that $\xi(t,x,y,x',y')$, the threat to an aircraft moving with velocity $v = (x',y')$ at a point $p = (x,y)$ at time t, is the solution to a PDE. We then wish to choose the flight path $p(t)$ that minimizes the total threat

$$J(p) = \int \xi(t,x,y,x',y')dt \qquad \text{(Eq 2-2)}$$

over the whole flight path.

A comment about the objective function, (Eq 2-2), is in order. A valid case can be made stating that this is not the most desirable objective. Indeed the "optimal" flight path $(x(t),y(t))$ could conceivably yield a curve in which the threat ξ, as a function of time is zero for most of the flight path but for a very small increment of time is so great as to be lethal. An alternative objective function might be the path that minimizes the largest value of ξ, ie

$$\min\{\max \xi(t)\}$$

If this were the objective then this model could still be used to generate data as a function of x, y, and t, but a different optimization scheme would have to be employed.

Pilots with different missions and aircraft will have differing, and subjective, opinions with respect to what constitutes a threat to their aircraft. A subsidiary objective was to utilize the expertise and opinion of the pilot with regard to the lethality of the different threat

systems. This not only increases the flexibility of the model in that it would necessitate allowing for different threat scenarios but it offers the man who actually has to fly the route some input as to what his "optimal" flight path is.

<u>Terminology</u>

A flight through a hostile air defense zone can certainly be considered a life threatening situation. But what does the term threat mean? For the purposes of this thesis the term threat will be left undefined and characteristics of threat will be hypothesized and eventually utilized in Ch 3 to derive the threat equation.

The primary hypothesis of this thesis is that threat, designated by the variable Q, changes with both time and position and that although threat is undefined we will be able to assign a scalar-valued function which describes at every point in space and at every time t the level of threat at that point in space-time. This function is a relative measure of threat and will be called the threat potential, Φ.

To understand this relationship between threat and threat potential it might be advantageous to consider the relationship between heat and temperature in thermodynamics. (See (Jakob, 1949:ch1-2)) The temperature of a body is a relative (depending on which temperature scale used) measure of the amount of heat in a body. In addition the amount of heat in a body changes with respect to both position and time. Since the threat potential is a measure of how "hot"

the situation is then the unit of threat potential will be designated a "degree".

Threat sources are "...those elements of a man made environment designed to reduce the ability of an aircraft to perform mission-related functions by inflicting damaging effects, forcing undesirable maneuvers, or degrading system effectiveness" (Ball, 1985:68). There are two types of threat sources: terminal and nonterminal. A nonterminal threat source does not have the ability to inflict damage. Examples of these are early warning radars or communications networks. A terminal threat source can cause damage. Guns, missiles, airborne interceptors and directed energy devices are all examples of terminal threat sources. (Ball, 1985:71-73)

A threat source function is defined to be a mathematical equation which describes how the threat is generated as a function of position within the air defense zone or time. If for example the aircraft entered the zone when the threat sources were not at full battle stations then it is conceivable that the threat potential, although low at first, would increase as a function of time as the threat sources became operational.

A threat sink function is a mathematical equation which describes how the threat is disposed of as a function of position and time. Offensive operations against a threat source could be considered as a threat sink function as long as it were possible to describe the operation mathematically.

The effectiveness of jamming a radar as a function of range is the most obvious.

Since survivability is a function of the hostile environment a knowledge of that environment is needed. An assumption will be made that the pilot has perfect knowledge of the different systems he is flying against and where those systems are located. In addition to this the pilot will be able to rank order the threat sources according to their lethality against his aircraft for the mission he is performing. Once the pilot has sorted the threat sources in order of increasing lethality he must assign a utility function to each source. This utility function is a relative measure of the lethality of each threat source according to the opinion of the pilot, based on the assumption of constant speed and constant altitude.

Let this utility function be called the "source strength" of the threat source, designated by "m", with units of "firepower". The use of the word firepower does not imply that only terminal threat sources must be rated. As an example let the air defense zone consist of 1 early warning radar, 1 AAA battery, and 1 SAM site and assume that the pilot has ordered the threat sources in order of increasing lethality. Then he might assign to the radar a source strength of 1 firepower (fp), to the AAA battery a strength of 10 fp, and to the SAM site a strength of 20 fp. In effect, the pilot is declaring that for a constant speed and altitude, the SAM site is 20 times more lethal than the

radar. Since 600 fp of source strength is more lethal concentrated in a 10 square mile area than in in 100 square mile area let σ be source strength per unit area.

Consider two air defense zones, each with the responsibility for defending the same amount of land area. In the first air defense zone there are 50 SAM sites and 100 AAA batteries. In the second zone there are 2 SAM sites and 4 AAA batteries. What would the effect be on the threat potential at any point if one more SAM site were added to each zone? Assume that all the SAM sites and AAA batteries are of the same type. Let the pilot give a rating of 10 fp to the SAM site and 1 fp to the AAA battery. The total source strength of the first zone is 600 fp. Adding one more SAM site increases the total source strength of the zone by 10 fp, or about 1.6%. Adding 1 SAM site to the second zone causes the total source strength of that zone to increase from 24 to 34 fp, or about 42%.

To account for this decreasing total marginal source strength let the variable c designate the specific threat capacity of the zone. Define c to be

$$c = (1/m) \cdot dQ/d\xi \qquad \text{(Eq 2-3)}$$

Another interpretation of the specific threat capacity is that the amount of threat needed, dQ, to raise the threat potential by $d\xi$ in an air defense zone with total source strength m is

$$dQ = mc\,d\xi$$

Or, in terms of source strength per unit area

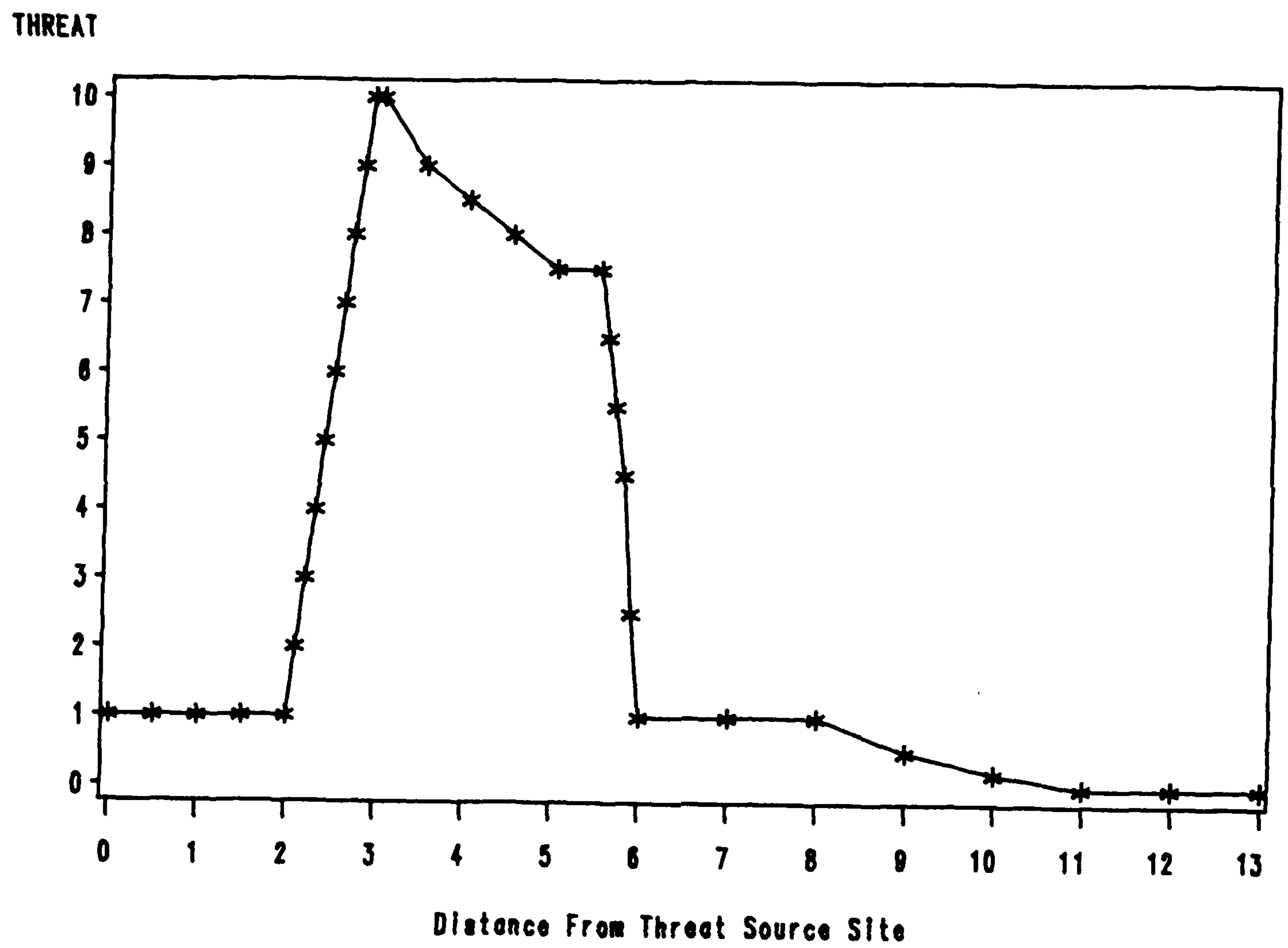
$$dQ = \sigma A c d\xi$$

where A is the area covered by the defenses.

Threat Relative to Position

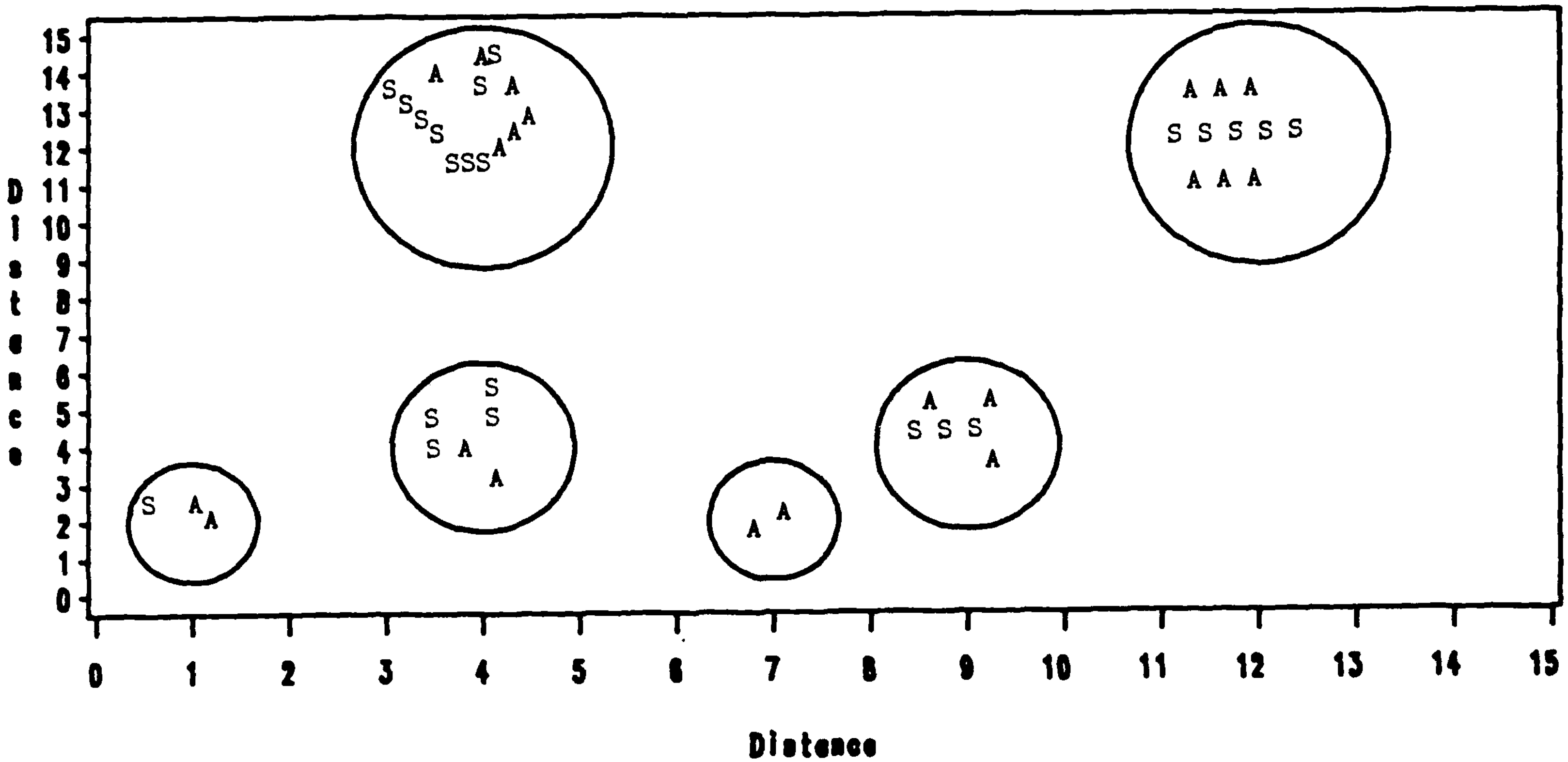
Probably the most difficult aspect of describing the threat to an aircraft in a one-on-many model is the description of threat as a function of position in the zone. If the speed and altitude are held constant then at the very least the threat depends on where the aircraft is relative to the different threat sources. In addition, certain areas of the zone will have overlapping coverage.

The objective then is to describe the threat to an aircraft as a function of position from the threat source in such a way that the effect of adding the threat from different threat sources can be accounted for. Consider the threat to an aircraft as a function of ground range from a generic threat source. One possible function is shown in Fig 2-1. In the immediate vicinity of the threat source the threat is negligible due to the fact that the aircraft is so close to the threat source as to render it almost useless. Of course this "dead zone" varies with different types of threat sources. A AAA battery might have almost no dead zone while certain SAM sites will be unable to engage for any distances less than 500 feet.



2-1 Hypothesized Threat Potential
as a function of distance from threat source

It seems reasonable to assume that the threat as a function of range from a threat source can be approximated as an m/r^2 relationship for all sources except perhaps a SAM. To overcome this difficulty and make the m/r^2 relationship more indicative of a true threat function what was done was to consider an array of threat sources as a single point source centered in the middle of the array. When combining threat sources it is not necessary to group the sources according to type of threat source. Although the combining of threat sources is arbitrary, a plausible heuristic argument for combining the sources according to geographic location is as follows. Since the threat is being approximated as an m/r^2 relationship the area close to the source should have a high threat level. Since there is a dead zone around any single threat source there would also be a dead zone around a group of homogeneous threat sources so long as the sources were close together. If however there were different threat sources within the grouping their dead zones would, in effect, cancel because one site's dead zone would be covered by another site. Fig 2-2 is an example of the different threat sources grouped as a point source. The SAM sites provide the coverage at greater distances and the AAA batteries cover the dead zone of the SAM sites as well as each others'. The source strength of the point source could be just the sum of the source strengths of the individual

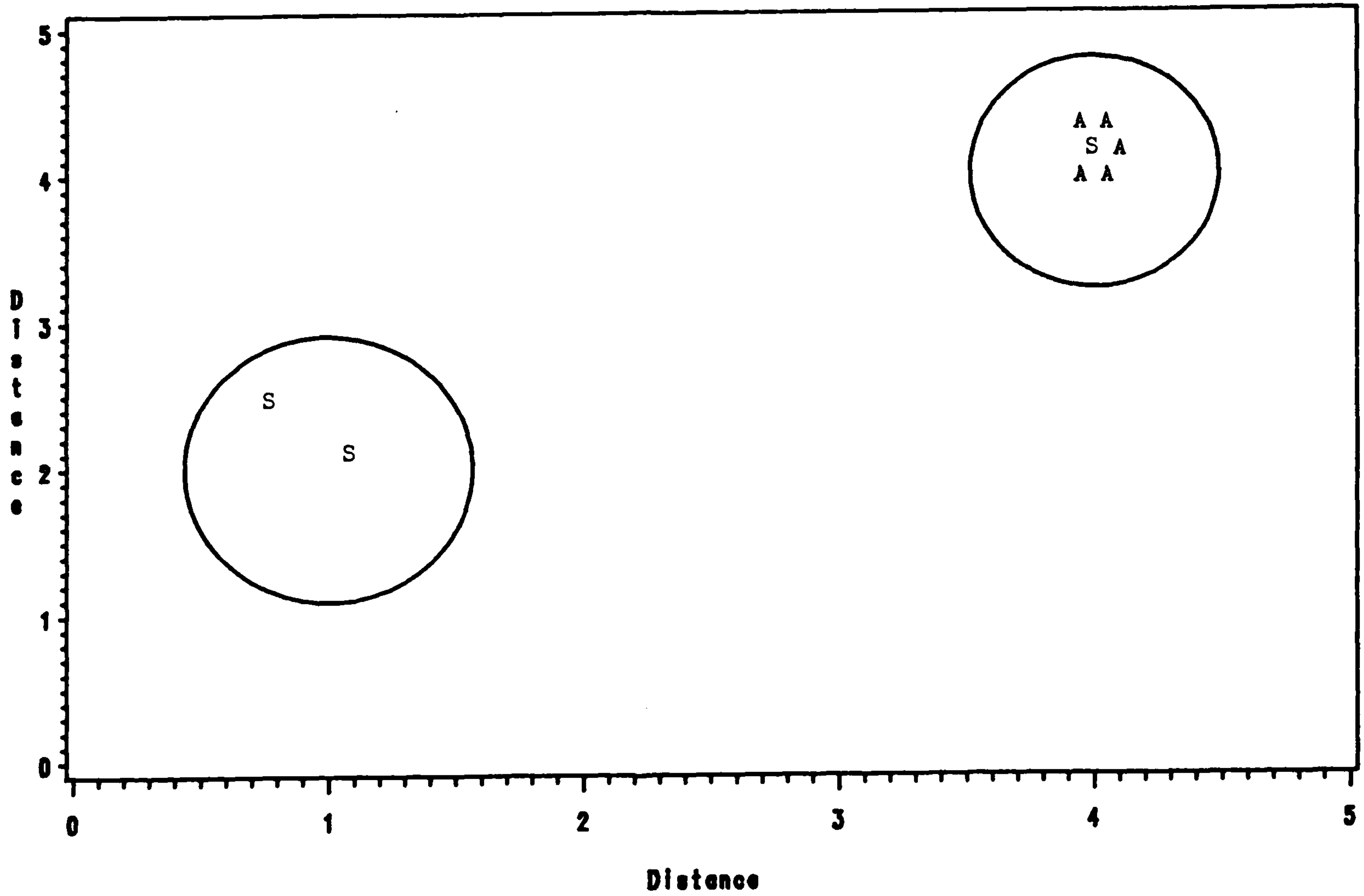


2-2 Threat Sources Considered as Point Sources

sources, or it could be assigned its own source strength rating by the pilot. For instance again let a SAM site have a source strength of 10 fp and a AAA battery a source strength of 1 fp and consider an air defense zone that consists of 3 SAM sites and 5 AAA batteries that are combined as in Fig 2-3. Which of these point sources is really the most lethal, the two SAM sites with a strength of 20 fp or the one SAM site and 5 AAA batteries with a strength of 15 fp? The answer is up to the pilot.

Conclusion

In this chapter previous uses of partial differential equations to describe systems as presented as well as objectives stated. In addition terms were defined and the threat as a function of position from the threat source was hypothesized as a m/r^2 relationship.



2-3 Hypothetical Threat Scenario

III. Model Presentation

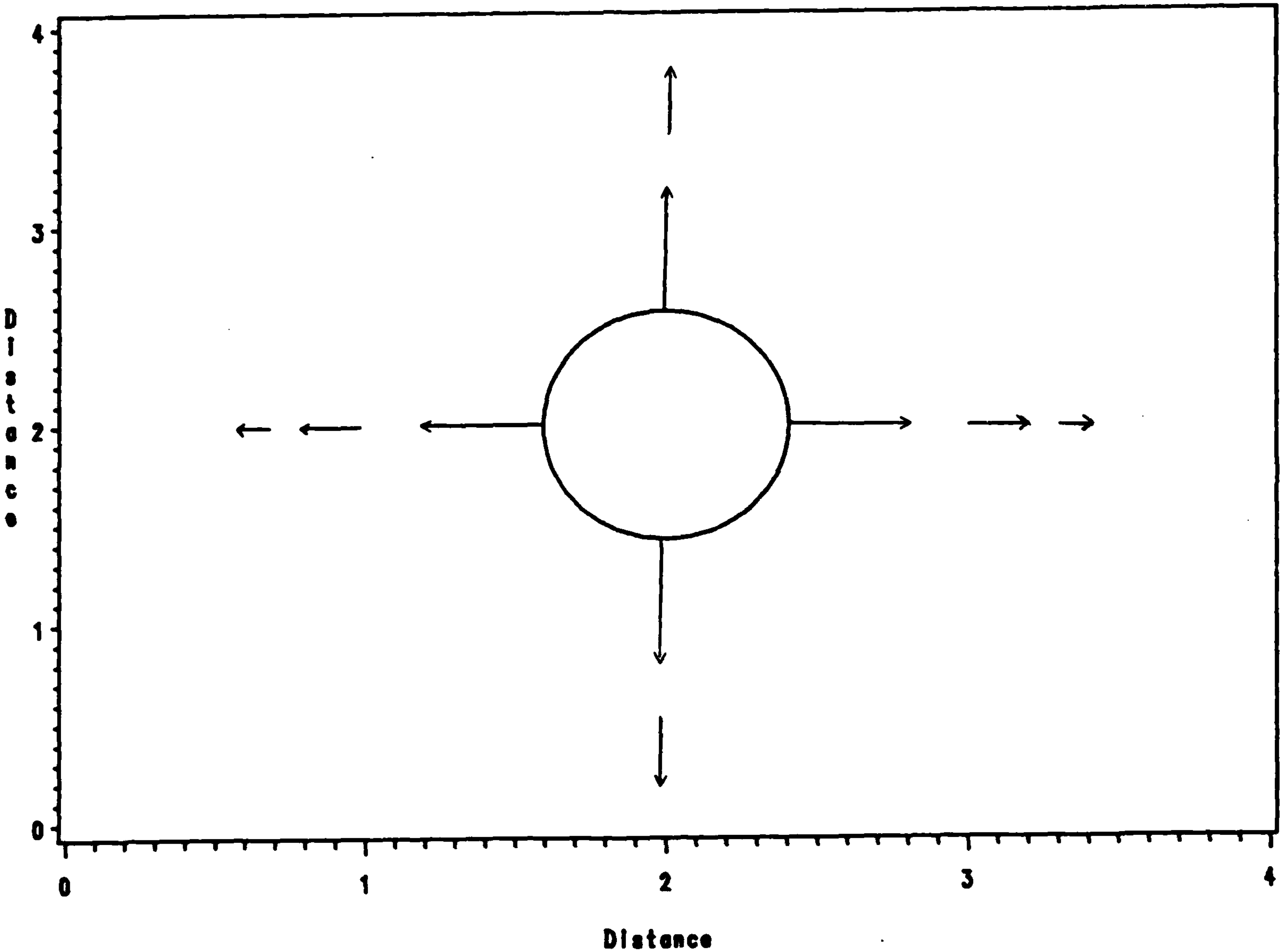
Further Definitions

Before preceding with the derivation of the threat equation three more terms must be defined. The terms that were presented in chapter 2 described the threat as a function of position. In addition, the size and hostility of the air defense zone were described in terms of the specific threat capacity and the source strength. The main hypothesis of this thesis, however, is that the threat changes with both time and position. What needs to be accomplished then is to define terms that present threat as a function of time and then relate the different terms.

Figure 3-1 is a graphic representation of the $1/r^2$ relationship hypothesized in chapter 2 that describes the threat as a function of position from the threat source with the threat vector directed radially outwards from the source. Those familiar with electrostatics will recognize the figure as the two dimensional electric field about a point charge. Abstractly, one can think of threat as "emanating" from the threat source. Let the vector H, the threat flux vector, be the rate of change of threat with respect to time,

$$H = dQ/dt$$

The magnitude and direction of H will be discussed presently.



3-1 Threat Source Strength

In chapter 2 the threat source and threat sink functions were defined as mathematical functions which described how threat was created and destroyed as a function of position and time. Let G be the total threat generation function which is just the addition of the threat source function and the threat sink function. That is

threat generated = threat created + threat destroyed

where threat destroyed is a negative function.

The final term to be defined is threat concentration. Choose an arbitrary volume of space anywhere in the air defense zone and let the threat concentration, ϵ, of that volume be the threat per unit volume that arises due to the threat flux.

Derivation of the Threat Equation

The presentation of the threat equation is based on the method for deriving governing equations for substance transfer found in (Aris, 1978:41-42).

Consider an aircraft with speed U performing a mission in a threat field. Construct a volume of arbitrary size and position in the threat field and denote the volume by V with boundary S. The rate at which the threat is entering the volume through the boundary S is proportional to the surface area, the threat flux vector H, and the angle which the threat flux vector makes with the outward drawn normal to the box. This is consistent with derivations of the heat equation

found in (Jakob, 1949:Ch 1). Figure 3-2 graphically presents the relationship (in two dimensions) between the different vectors where N is the outward normal vector.

The component of H parallel to N is $-H\cos\theta$ where the minus sign indicates that the threat is flowing into the volume. Let N be a unit vector. Then the component of H parallel to N can be written as

$$-H\cos\theta = -H \cdot N$$

where $\cdot$ indicates the scalar dot product of two vectors. Therefore the rate of threat entering the volume through the boundary of the volume is

$$-\int_S H \cdot N \, dS \qquad \text{(Eq 3-1)}$$

This integral can be transformed into a volume integral by invoking the Divergence Theorem which states that

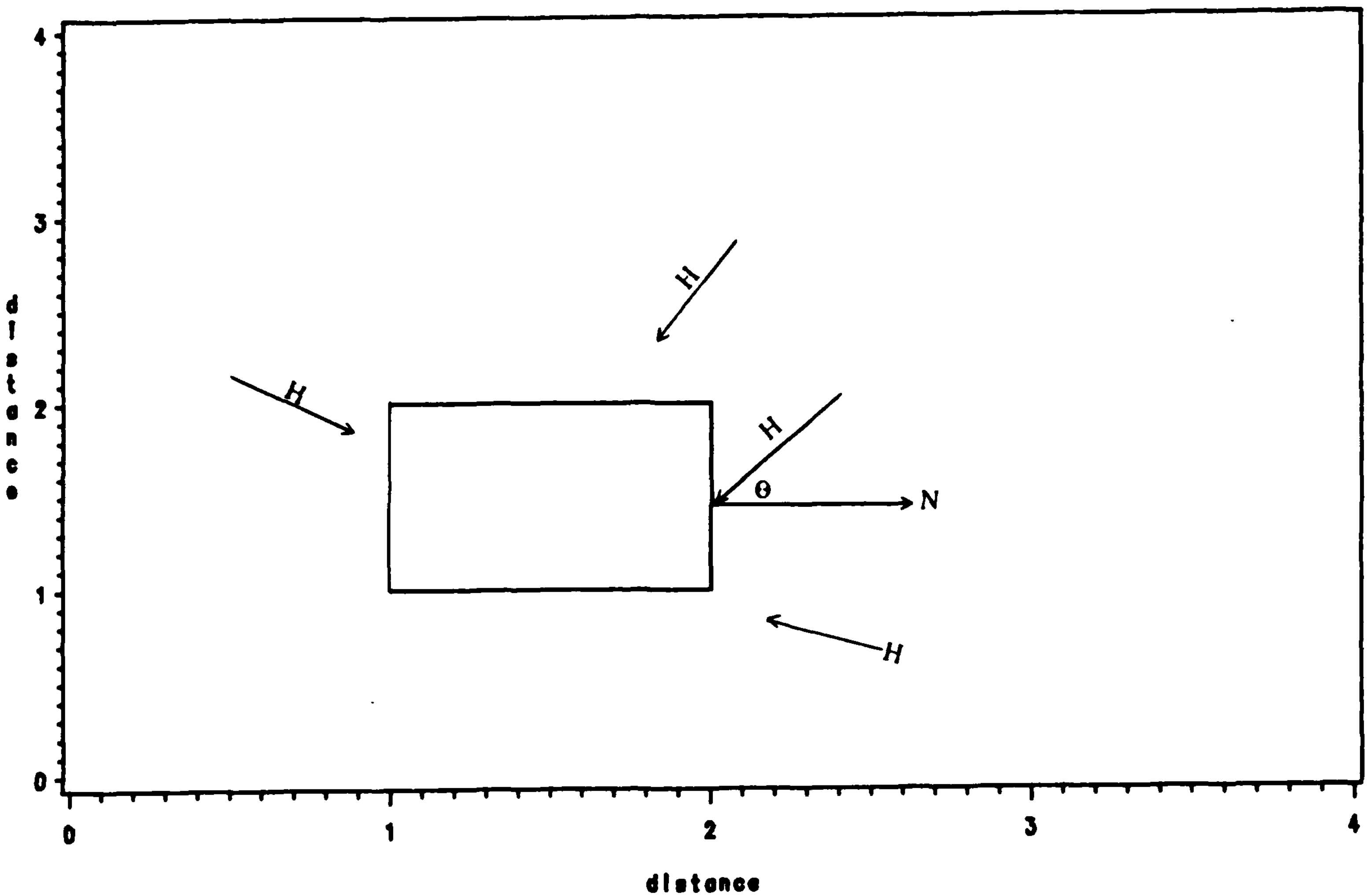
$$-\int_S H \cdot N \, dS = -\int_V \text{div } H \, dV$$

where div H is the scalar dot product of H with the vector differential operator del

$$\text{del} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$$

Let G denote the rate at which the threat is generated per unit volume. Then the rate of threat generated in the volume V is simply

$$\int_V G \, dV \qquad \text{(Eq 3-2)}$$



3-2 Relationship Between H and N

Again, the threat generated is simply the sum of a threat source and a threat sink function.

So far there is threat entering the volume through the boundaries and threat being generated within the volume so that inside the volume there is a concentration of threat. Remembering that the threat per unit volume is denoted by ϵ then the rate at which threat is maintained in V is

$$D/DT\left(\int_V \epsilon \, dV\right) \qquad \text{(Eq 3-3)}$$

where D/DT is the full derivative, ie

$$D/DT = \partial/\partial t + v \cdot \text{del}$$

where v is the speed of a moving volume, not the aircraft. Equation 3-3 simply states that the concentration of threat in the volume is not only a function of time but also depends on the fact that the volume is moving. For the analysis in Ch 4 it is assumed that the volume is stationary so $v = 0$. By invoking the Reynolds' Transport theorem (Eq 3-3) becomes

$$D/DT\left(\int_V \epsilon \, dV\right) = \int_V (\partial\epsilon/\partial t) \, dV + \int_S \epsilon \, v \cdot N \, dS \qquad \text{(Eq 3-4)}$$

where N is again the unit outward normal vector. By again invoking the divergence theorem on the last integral on the right in Eq 3-4 one arrives at

$$\int_S \epsilon v \cdot N \, dS = \int_V \text{div}(\epsilon v) \, dV$$

Therefore the rate of threat stored in the volume is

$$\int_V [\partial\epsilon/\partial t + \text{div}(\epsilon v)] \, dV \qquad \text{(Eq 3-5)}$$

If the conservation of threat is postulated then the equation which results is

$$\begin{array}{c} \text{Rate of threat} \\ \text{stored in} \\ V \end{array} = \begin{array}{c} \text{Rate of threat} \\ \text{entering and exiting} \\ V \end{array} + \begin{array}{c} \text{Rate of threat} \\ \text{generated in} \\ V \end{array}$$

which can be written mathematically as

$$\int_V [\partial\epsilon/\partial t + \mathrm{div}(\epsilon v)]\, dV = -\int_V \mathrm{div}\, H\, dV + \int_V G\, dV \qquad \text{(Eq 3-6)}$$

Since the size of volume was arbitrary let it be small enough so that the integral sign can be ignored. Then

$$\partial\epsilon/\partial t + \mathrm{div}(\epsilon v) + \mathrm{div}\, H = G \qquad \text{(Eq 3-7)}$$

A vector differential identity states that

$$\mathrm{div}(\epsilon v) = \epsilon\, \mathrm{div}\, v + v \cdot \mathrm{grad}\, \epsilon$$

where grad is the gradient operator. Equation 3-7 can then be written as

$$\partial\epsilon/\partial t + \epsilon\, \mathrm{div}\, v + v \cdot \mathrm{grad}\, \epsilon + \mathrm{div}\, H = G$$

The following assumptions are now made. First, that there is a relationship between the scalar threat potential ξ, the threat flux vector H and the threat concentration ϵ. Second, assume that the relationship between ξ and ϵ is linear

$$\epsilon = \sigma c \xi \qquad \text{(Eq 3-8)}$$

where σ is the threat strength per unit area and c is the specific threat capacity of the zone. Intuitively this just says that has the threat concentration in the volume increases then the threat potential also increases. For the case of a constant altitude which is considered in this thesis the three spatial dimensions reduce to only two and

equation 3-8 is correct. For the more general problem with varying altitude equation 3-8 is not correct, in fact the units do not match. In future work a new equation comparable to equation 3-8 would have to be derived. The threat source strength per unit area would then have to be considered as a threat source strength per unit volume, in which case the units of equation 3-8 would be consistent.

The final assumption is that the threat flux vector can be written as the gradient of the threat potential:

$$H = -wU\,\mathrm{grad}\,\Phi \qquad (w>0) \qquad \text{(Eq 3-9)}$$

where U is the speed of the aircraft and w is a constant of proportionality. To simplify let $k = wU$ with units of threat/sec·deg where the units of distance in U cancelled with the units of distance in grad Φ. The variable k will be called the threat conductivity. The threat conductivity is a measure of how quickly the threat potential can change as a function of time. The threat conductivity variable is related to the speed of an attacking aircraft and since that speed is assumed constant so will the threat conductivity. It is assumed that the threat potential at any point in space at any time t is also a function of the aircraft speed and that the threat potential decreases with increasing speed. The negative sign in equation 3-9 indicates the direction of H. Basically equation 3-9 states that threat flows from a point of higher threat potential to a point of lower threat potential and the rate at which it flows is directly

proportional to the speed of the aircraft. If the speed of the aircraft is relatively slow then points of higher threat potential will remain points of higher potential longer than if the aircraft was moving faster. The faster the aircraft flies the faster the threat potential of the entire air defense zone approaches a constant.

The threat equation now becomes

$$(\sigma c)\partial\xi/\partial t + \sigma c\xi \operatorname{div} v + v\cdot\sigma c \operatorname{grad} \xi - \operatorname{div}(k \operatorname{grad} \xi) = G \qquad \text{(Eq 3-10)}$$

Realizing that

$$\operatorname{div}(k \operatorname{grad} \xi) = k\Delta\xi \quad \text{(k held constant)}$$

where $\Delta\xi = \partial^2\xi/\partial x^2 + \partial^2\xi/\partial y^2 + \partial^2\xi/\partial z^2$ is the Laplacian of ξ yields

$$(\sigma c)\partial\xi/\partial t + \sigma c\xi \operatorname{div} v + v\cdot\sigma c \operatorname{grad} \xi - k\Delta\xi = G$$

Let $a^2 = k/\sigma c$ and let $v = 0$. Then the final form of the threat equation is

$$\partial\xi/\partial t = a^2\Delta\xi + G/\sigma c \qquad \text{(Eq 3-11)}$$

Notice that a^2 has units of $(\text{distance})^2/\text{time}$ so that equation 3-11 is consistent with respect to dimensions.

Conclusion

In this chapter more terms were defined and the threat equation was presented. The threat equation was presented in the context of a "substance", called threat, and the movement of that substance with respect to position and time. The derivation of the threat equation closely parallels that of

the heat equation in the science of heat transfer. This relationship is explicitly shown below.

Symbol	Heat Model	Threat Model
Q	Heat	Threat
H	Heat Rate	Threat Rate
Φ	Temperature	Threat Potential
m	Mass (grams)	Threat Strength (fp)
c	Specific Heat	Threat Capacity
k	Conductivity	Threat Conductivity

IV. Analysis

Introduction

This chapter consists of a brief overview of the methodology proposed for obtaining an optimal flight path through a threat field where the threat potential changes according to the threat equation. In addition, under certain assumptions, optimal flight paths are obtained using this methodology. For extremely simple forms of the threat equation, closed form solutions may be obtained. These closed form solutions, which would describe the threat as a function of position and time, would then be used as the integrand in the calculus of variations problem referred to in chapter 1. For the more complicated problems considered in this thesis this was not accomplished due to the somewhat contrived scenarios needed to generate closed form solutions to the diffusion equation, which is the name given to Eq 3-11 in the science of heat and mass transfer. For many of the examples in this science the symmetry of the problem (infinite rod, infinite cylinder, etc) is taken advantage of to obtain the solution. The symmetrical examples mentioned above, as well as others, will have significant meaning in heat transfer but can not be easily extended to explain the threat potential in a changing threat field. Rarely is there any symmetry in an air defense zone. What was done then was to numerically solve the threat equation, hypothesize a closed form equation which would be fit to the data, and then find the coefficients of

the independent terms of the closed form which minimized the sum of the deviations squared. This best fit function was then used as the integrand in the calculus of variations problem.

The analysis was performed in both one and two spatial dimensions. For the two dimensional case both time varying and time invariant scenarios were considered. Closed form solutions for the one dimensional case were obtained. Although the flight path for the one dimensional case is already known the justification for performing this analysis was twofold. First, to exhibit the dependence of the threat potential on the aircraft speed, and secondly to hypothesize the form of the two dimensional solution used to fit the data generated by the two dimensional threat equation. The reason for obtaining a flight path through a time invariant threat potential was to compare the calculus of variations flight path to both a dynamic programming flight path and the shortest path algorithm. (The "shortest path" meaning the path of least threat.)

Overview of Methodology

Under the assumption that all threat sources and their positions are known, as well as a constant aircraft speed, altitude, and pilot ratings of the threat sources, the algorithm for determining the optimal flight path is as follows:

Step 1: Establish initial and boundary conditions from

pilot ratings to be used in solving the threat equation.

Step 2: Identify threat generation function G.

Step 3: Solve threat equation with resulting solution a function of time and space, $\xi(t,x,y)$. If solution is in closed form then proceed to Step 5.

Step 4: If solution from Step 3 is obtained using numerical techniques then hypothesize a threat function and fit the data so as to minimize the sum of the deviations squared.

Step 5: Using the closed form solution obtained from Step 3 or the least squares fit obtained in Step 4 as the integrand for a calculus of variations problem solve the resulting differential (ie- Euler) equations to obtain the path of least threat.

The analysis in this chapter was conducted under the assumption that steps one and two had been performed. It was considered more important to study the behavior of the solutions to the threat equation and the resulting optimal flight paths than to spend resources investigating pilot ratings of specific threat sources. Since the main objective was to formulate a methodology for validating the threat equation, assuming steps one and two was not considered critical.

Examples of Model Operation

As was stated earlier the threat equation is known as the diffusion equation in the field of heat and mass transfer. As such, the solution to the diffusion equation predicts, for example, the temperature of a body as a function of position and time with known initial and boundary temperatures. For instance, if the initial temperature of a square plate with negligible thickness is 30 degrees and the boundaries are maintained at 40 degrees, experiments indicate, and the heat equation predicts, that eventually the temperature of the plate will approach a constant relatively close to 40 degrees. This assumes that no heat is added to or removed from the plate. The plate simply gets hotter since the boundaries are maintained at that higher temperature. Conversely, if the boundaries are maintained at 30 degrees and the initial temperature is 40 degrees then eventually the plate will cool down to approximately 30 degrees. (Also assuming that no heat is added to or removed from the plate.)

When analyzing a changing threat potential in an air defense zone however there is no benefit of experiment except the undesirable observation of a downed aircraft. Intuition has to take the place of experiment. This dependence on intuition leads of course to varying interpretations when setting initial and boundary conditions as well as identifying a threat generation function.

Consider the following scenario. An aircraft enters an air defense zone in which the initial and boundary conditions, according to pilot inputs, has been specified. Therefore at $t = 0$ the threat potential at every point in the air defense zone is known. Assume that the aircraft has no offensive capability, that is, there will be no jamming and the aircraft will not fire at any of the threat sources. The sole objective was to go from a point on one side of the defense zone to a point on the other side of the air defense zone. Make the further assumption that the air defense zone has a central controller, someone who is able to communicate with and control the different threat sources and who, at the time of aircraft penetration, has all threat sources on alert. If all threat sources are at alert is it possible for the threat potential at all points in the zone to increase as time increases?

Any pilot who has operated in that environment will of course answer yes. Once a SAM gets airborne in many cases the threat has not only increased but is rapidly approaching infinity. But for this thesis the missile versus aircraft encounter is part of a "microscopic" model. The real question being asked is "Could the threat supplied by a threat source increase as a function of time if the threat source was on full alert?" If the threat source was not on alert then it is conceivable that the threat could increase as a function of time as the sources were becoming operational. But if the threat sources were already at full battle stations the

threat "emanating" from the threat source could be modeled as remaining constant or even decreasing, unless of course something were done by the controller to increase the threat.

The point to this discussion was to identify the problem of translating a real world experience into a mathematical equation while still retaining enough real world qualities to make the model intuitively appealing. Since the threat equation is just another name for the diffusion equation the solutions to the threat equation will behave exactly like the solution to the diffusion equation. That is if an initial threat potential of 40 degrees with the boundaries maintained at 30 degrees is specified then eventually the threat potential will level off to about 30 degrees. This assumes that no threat is "added" during the battle. Not only is it counter-intuitive to have the threat potential decrease since in general one expects the threat to get worse as one spends more time in the air defense zone but how is threat "added" to an air defense zone?

The threat equation does however have a mechanism for studying the threat under differing interpretations. This mechanism is the threat generation function. For instance, as was stated above intuition tells us that the longer an aircraft flies in an air defense zone the less chance the aircraft has of surviving. This intuition could be interpreted as the threat potential increasing with time uniformly over the whole air defense zone. If this were the case then the threat generation function would be specified

as some increasing function of time. This is what is also meant by "adding" threat. For the purposes of this thesis adding threat to a air defense zone does not necessarily mean that more threat sources have been added to the battle. Adding, or for that matter subtracting, threat is identical to specifying a nonzero threat generation function, and this is one of the areas where intuition must take the place of experiment.

The analysis in this chapter will be accomplished with the threat generation function set equal to zero. The interpretation given to this zero threat generation function is one of no offensive operations or jamming by the aircraft and the air defense zone at full alert with no capability for making the threat any worse than it will be at the start of the battle. This is admittedly somewhat contrived but it will become apparent that even in this simple scenario the mathematics can become somewhat involved.

<u>One Dimensional Motion</u>. (Farlow, 1982;ch5) For one dimensional motion the threat equation reduces to (with initial and boundary conditions specified):

$$\partial\Phi/\partial t = a^2 \partial^2 \Phi/\partial x^2 \qquad \text{(Eq 4-1)}$$

$$0 \le x \le 1, \quad t \le \infty$$

$$\Phi(0,t) = \Phi(1,t) = 0$$

$$\Phi(x,0) = f(x)$$

The solution to this problem will be accomplished by the method of separation of variables. Assume that the solution

$\xi(x,t)$ can be written as

$$\xi(x,t) = X(x)T(t)$$

After making this substitution and dividing both sides of Eq 4-1 by a^2XT we arrive at

$$(1/a^2T(t))(dT(t)/dt) = X''(x)/X(x) \qquad \text{(Eq 4-2)}$$

where $X''(x) = d^2X/dx^2$.

Since x and t are independent the only way for (Eq 4-2) to hold is for each side of equation 4-2 to equal a constant. Let this constant be $-p^2$. The result then are two ordinary differential equations.

$$dT(t)/dt = -(pa)^2T \qquad \text{(Eq 4-3)}$$

$$X''(x) = -p^2X(x) \qquad \text{(Eq 4-4)}$$

Equations 4-3 and 4-4 have as their solutions

$$T(t) = A\exp[-(pa)^2t] \qquad \text{(Eq 4-5)}$$

$$X(x) = B\sin(px) + C\cos(px) \qquad \text{(Eq 4-6)}$$

where A, B, and C are to be determined by the initial and boundary conditions. Therefore, the threat potential at any point x for any time t is

$$\xi(x,t) = \exp[-(pa)^2t](B\sin(px) + C\cos(px)) \qquad \text{(Eq 4-7)}$$

This is the general form of the solution to Eq 4-1 where the initial and boundary conditions have not yet been accounted for. In order to satisfy the boundary conditions given in Eq 4-1 we must have the constant $C = 0$ and $p = \pm\pi, \pm2\pi, \pm3\pi, \ldots$ Then for these particular boundary conditions the

threat is

$$\xi_n(x,t) =$$

$$A_n \exp[-(n\pi a)^2 t]\sin(n\pi x) \quad n = 1,2,\ldots \qquad \text{(Eq 4-8)}$$

These are the fundamental solutions (Farlow, 1982:37) of (Eq 4-1). These fundamental solutions are added in such a way that the initial conditions are satisfied. That is

$$f(x) = \sum A_n \sin(n\ x) \qquad 1 \le n < \infty$$

This is just the Fourier sine expansion of $f(x)$. The coefficient A_n is

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \qquad \text{(Eq 4-9)}$$

The final solution to Eq 4-1 is

$$\xi(x,t) = \sum \xi_n(x,t) \qquad 1 \le n < \infty$$

with A_n given as in Eq 4-9.

The observation to be made is that for fixed t and x the threat potential is a function of $a^2 = k/\sigma c$. The threat conductivity k is directly proportional to the aircraft speed v. As v gets larger so does a^2. As a^2 gets larger the threat potential, for fixed t and x, decreases. This is intuitively appealing since in general the faster one flies through an air defense zone the better the chances are of survival. Notice also that as σ, the threat source strength per unit area, increases so does the threat potential.

<u>Two Dimensional Motion</u>: <u>Time Independent Threat.</u> In order to compare the optimal flight path provided by the calculus of variations to optimal flight paths provided by

other optimization techniques the time dependence of the threat potential was initially ignored. The threat potential at discrete intervals was arbitrarily assigned as in Fig 4-1. The optimal path calculated using both dynamic programming (Przemieniecki, undated:ch6) and the shortest path algorithm (Hillier and Leibermann, 1980:ch6) is shown in Fig 4-2 along with the calculus of variations solution. The objective function for both the dynamic programming scheme and the shortest path algorithm was to minimize

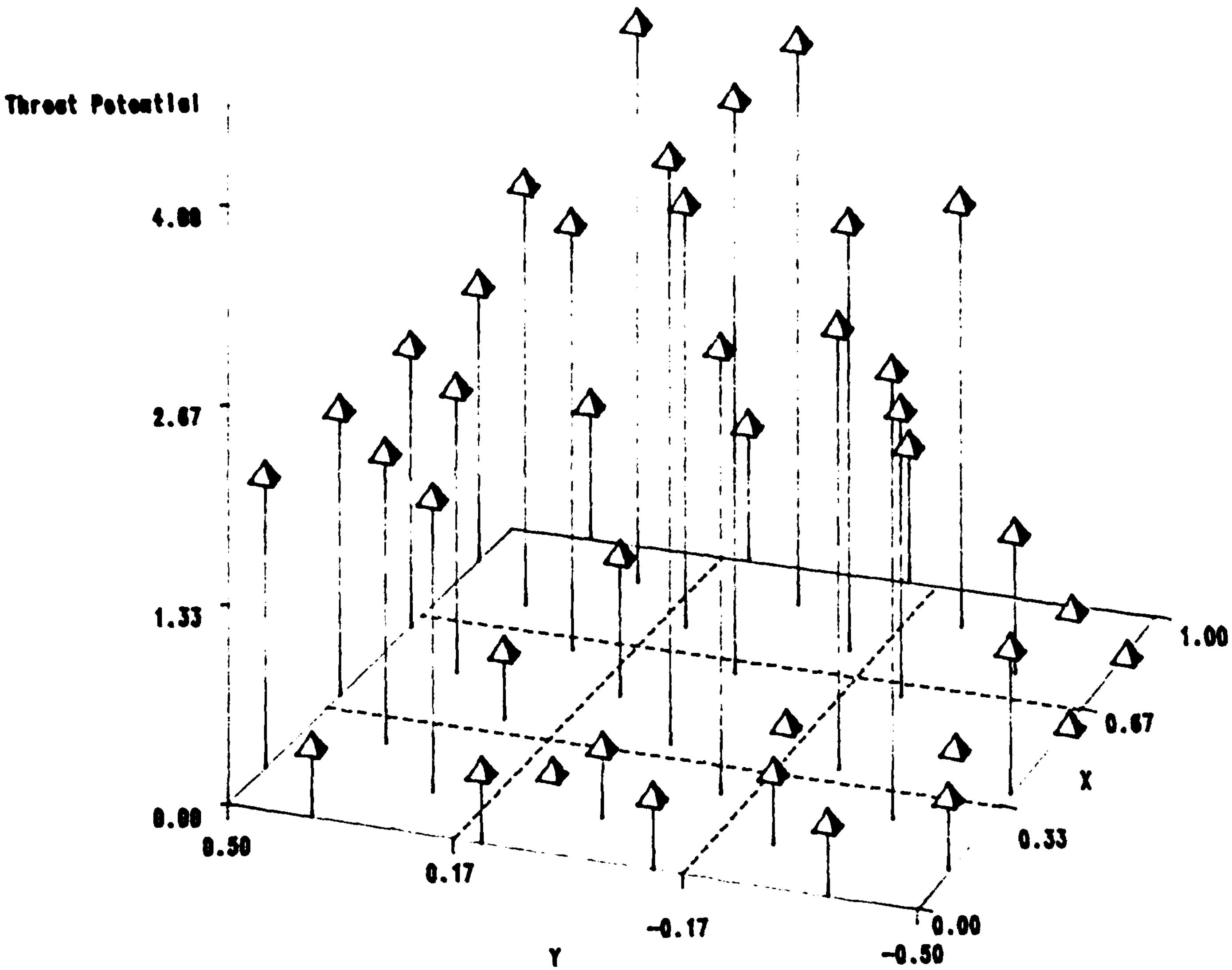
$$\xi = \sum \xi_{ij} \quad 0 \le i,j \le 4 \qquad \text{(Eq 4-10)}$$

Both dynamic programming and the shortest path algorithm predicted the same optimal path. This is the path that minimizes (Eq 4-10) provided no backtracking is allowed.

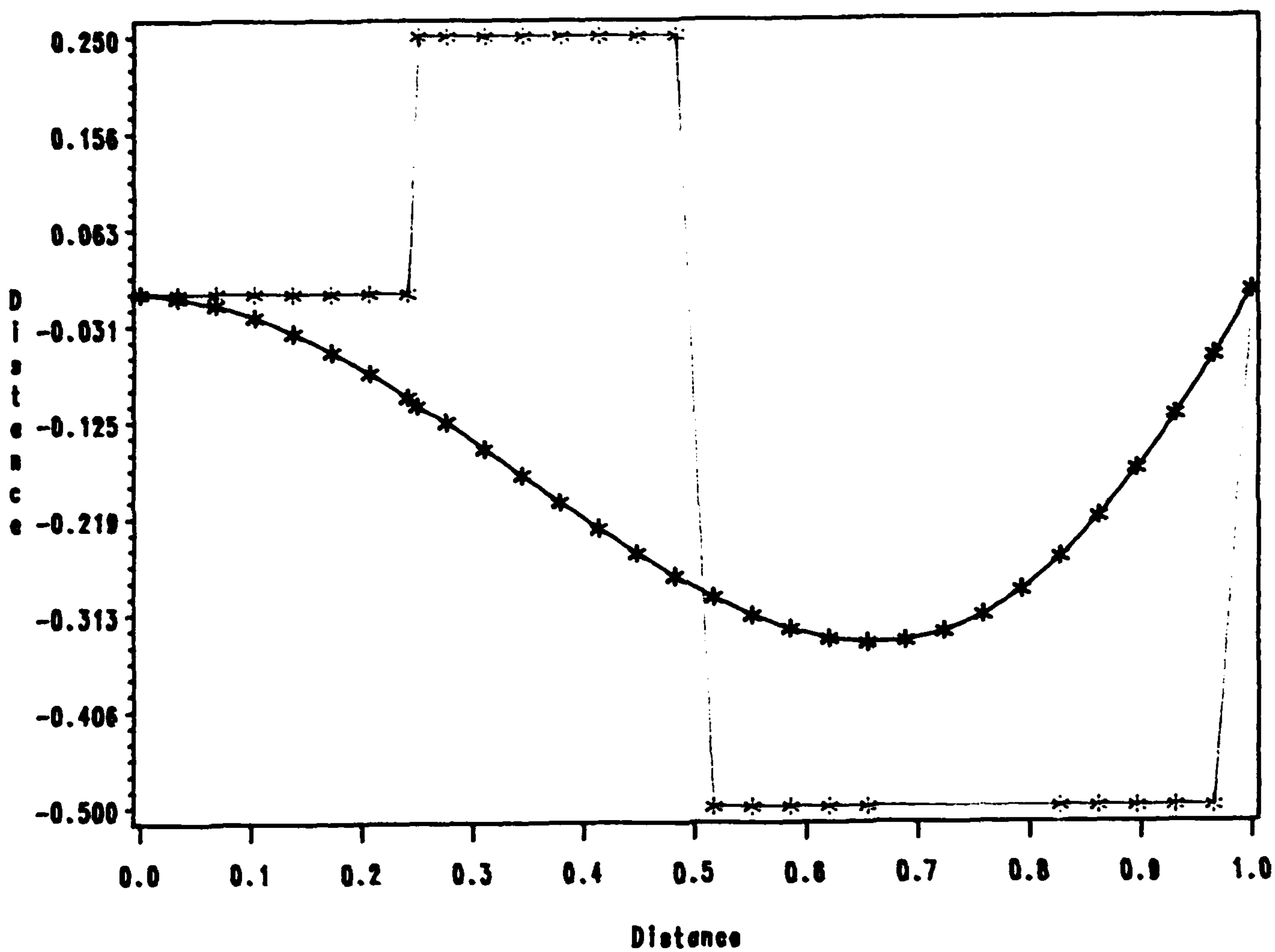
In order to utilize the calculus of variations a closed form for the threat potential was needed. A global fit of the data in Fig 4-1 was accomplished using the SAS Institute Inc. SAS/ETS system. The SAS/ETS procedure used was SYSNLIN, which fits data to a given nonlinear function. The form of the hypothesized threat function was

$$\begin{aligned} \xi(x,y) = {} & A30x^3 + A21x^2y + A12xy^2 + A03y^3 \\ & + A20x^2 + A11xy + A02y^2 \\ & + A10x + A01y + A00 \qquad \text{(Eq 4-11)} \end{aligned}$$

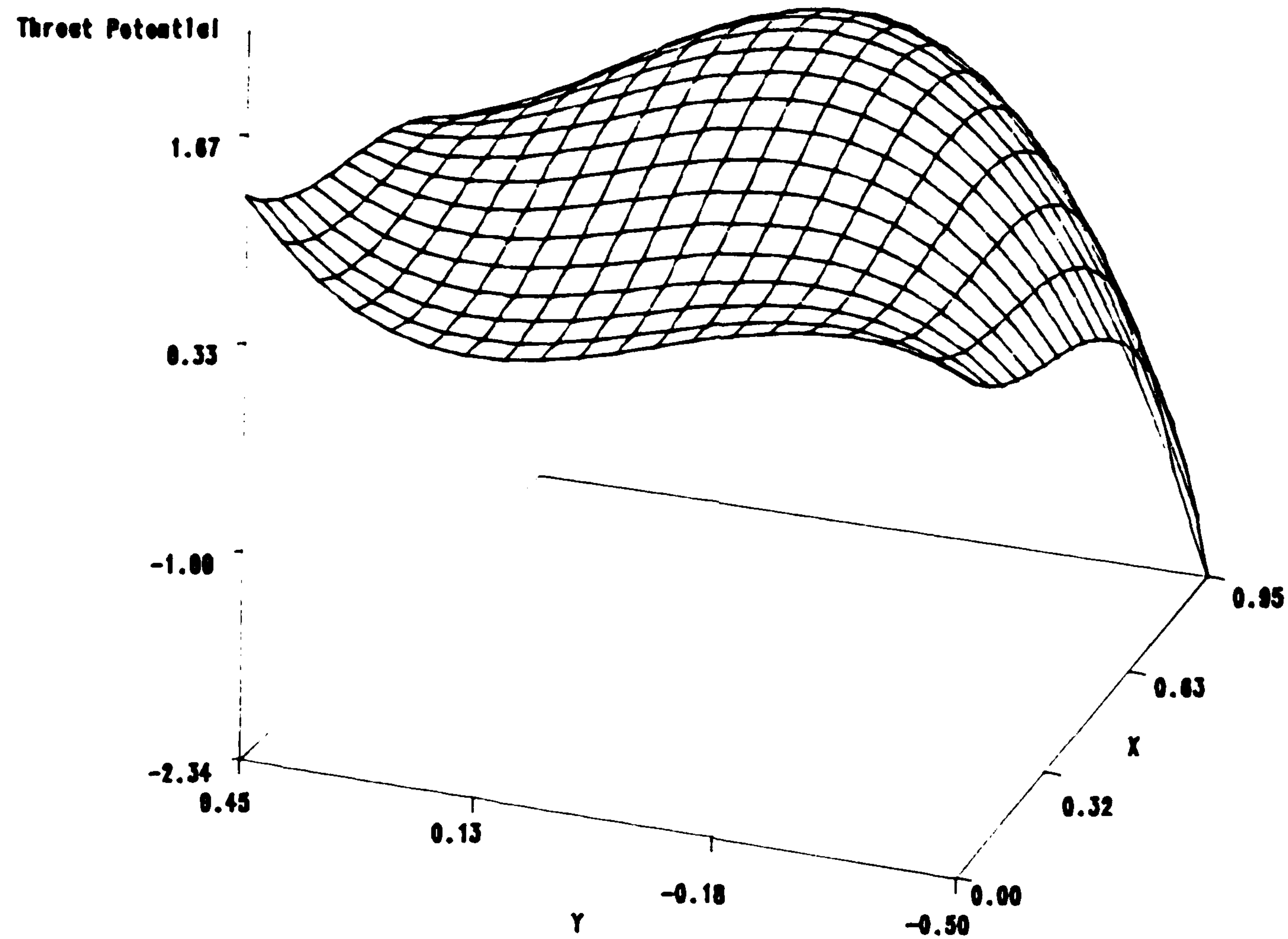
Values for the coefficients Aij were obtained and are listed in the appendix. At this point many of the pit falls associated with nonlinear programming were realized. Figure 4-1 is the scatter plot of the actual data while Fig 4-3 is a



4-1 Scatter Plot of Threat Data



4-2 Optimal Paths: Time Independent
Threat Potential



4-3 Least Squares Fit of Threat Data

plot of (Eq 4-11) with the actual coefficients that were obtained by SAS. As can be seen by the two graphs Fig 4-3 is a relatively smooth function while Fig 4-1 has some large peaks and valleys. Large differences in threat potential between closely spaced points were lost in the global fit. In addition, the SAS/ETS system computes the probability of each of the Aij's being zero. Only those coefficients with a probability less than 0.5 were used in equation 4-11 and in the determination of the optimal flight path.

The integral which was minimized was

$$J = \int \xi(x,y)\, ds \qquad \text{(Eq 4-12)}$$

If no backtracking is allowed then y is a function of x so

$$ds = (1 + y'^2)^{1/2} dx$$

where $y' = dy/dx$. Then (Eq 4-12) becomes

$$J = \int_0^1 \xi(x,y)(1+y'^2)^{1/2} dx$$

Letting $F(x,y,y') = \xi(x,y)(1+y'^2)^{1/2}$ the theory of the calculus of variations states that in order to minimize Eq 4-12 $F(x,y,y')$ must satisfy Euler's equation

$$d/dx(F_{y'}) - F_y = 0$$

where the subscript F_i indicates taking the partial derivative of F with respect to the variable i. Then

$$F_y = \xi_y(1+y'^2)^{1/2}$$

$$F_{y'} = \xi y'(1+y'^2)^{-1/2}$$

and

$$d/dx(F_{y'}) = (1+y'^2)^{-1/2}[\xi_x y' + \xi_y y'^2 + \xi y''(1/(1+y'^2))]$$

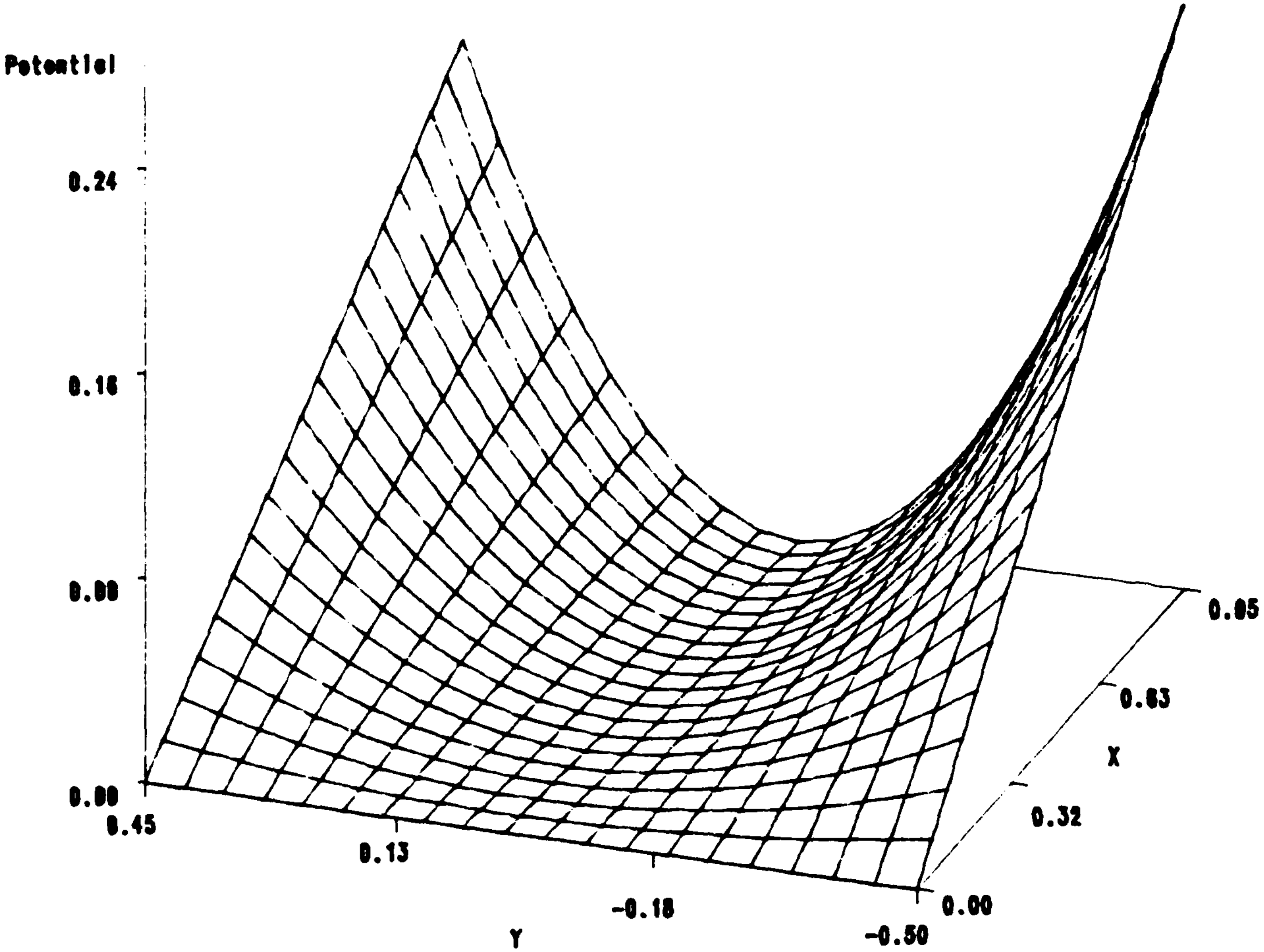
Upon simplifying, Euler's equation becomes

$$\xi_x y'(1+y'^2) + \xi y'' = \xi_y (1+y'^2) \qquad \text{(Eq 4-13)}$$

This second order ordinary differential equation can then be reduced to a system of first order differential equations by making the substitution $y_1 = y$ and $y_2 = y_1'$. Therefore Eq 4-13 becomes

$$y_1' = y_2$$
$$y_2' = (1+y_2^2)[\xi_y - \xi_x y_2]/\xi$$

Since the end points of the flight path are known, the IMSL routine DVCPR, which solves two point boundary value problems was used to solve the system. A program was written to test the DVCPR routine with a given threat potential shown in Fig 4-4. The threat potential was $\xi(x,y) = xy^2$. With this threat potential contour the flight path should be directly along the x axis. This flight path was indeed obtained. An attempt was then made to solve the system associated with (Eq 4-13). This attempt was unsuccessful because the solution technique failed to converge. A simplification, $y_2^2 = 0$, was made. This simplification was then made in the test program to see if the flight path would remain along the x axis, which it did. The simplified system associated with (Eq 4-13) did converge and the flight path is the curved path shown in Fig 4-2. One drawback of the DVCPR routine was the need for an initial



4-4 Test Threat Potential: $\Phi = xy^2$

guess for the flight path. At first an initial guess of y being linear with x was used and this also failed to converge. Next, an initial guess of $y = .5x^2$ was assumed and this did converge.

The only major difference in the flight paths of the different optimization schemes is the path taken between x = 0.2 and x = 0.5. This discrepancy can be attributed to a number of things. First, both dynamic programming and the shortest path algorithm are dependent on the number of steps available, ie, the number of grid points. As the number of grid points, and therefore the number of specified threat potentials at those grid points, increases the dynamic programming and shortest path algorithm flight path could possibly approach a smoother curve (This of course depends on the data at those grid points). Secondly, as was previously stated, the closed form of the threat potential was obtained using a global fit with the result that large differences in threat potential between closely spaced points were lost. Finally, the introduction of a simplification so that the system of differential equations could be solved will also introduce some error. The common characteristic of the two solutions was the fact that both solutions avoided the large threat potentials along the x axis and the paths followed points of lower threat potential in the negative y plane.

<u>Time Dependent Threat</u>. An attempt was made to introduce time as a factor in the determination of an optimal flight

path. Assume for the present that a particular battle scenario has been agreed upon and a closed form solution $\xi(x,y,t)$ for the threat is given (and nonlinear in not only x and y as in the time independent case but also nonlinear in t). Then the integral which was minimized was

$$J = \int_0^{t_f} \xi(x,y,t)\ (1+x'^2+y'^2)^{1/2}\ dt \qquad \text{(Eq 4-14)}$$

where now the primes indicate taking the derivative with respect to t. Equation 4-14 is a natural extension of equation 4-12 to three dimensions where the 1 under the radical sign has units of velocity squared. The term under the radical sign is arc length in x-y-t space which is acceptable only when x,y, and t have the same dimensions. In order for this to occur there must be a constant of proportionality, that is

$$(ds)^2 = (udt)^2 + (dx)^2 + (dy)^2$$

where u has units of velocity. For these examples u = 1. The arc length ds in this space should not be confused with the path length which would be flown in the real world. An understanding of this u term can be gained when one considers the case of u = 0. For this case also assume that x' and y' are also zero, so that, in effect, the aircraft is hovering at one point in the x-y plane (ie- a helicopter). Equation 4-14 is then zero which says that there is no threat accumulating when the aircraft is hovering. This is of course not the case since a stationary aircraft in an air defense

zone means instant death, hence the inclusion of a nonzero u term.

In order to simplify notation let

$$L(x',y') = (1+x'^2+y'^2)^{1/2}$$

$$F(t,x,y,x',y') = \xi(x,y,t)L(x',y')$$

Now, in order to minimize equation 4-14, a system of Euler's equations must be satisfied

$$d/dt(F_{x'}) - F_x = 0 \qquad \text{(Eq 4-15)}$$

$$d/dt(F_{y'}) - F_y = 0 \qquad \text{(Eq 4-16)}$$

where again the subscripts indicate taking the partial derivative of F with respect to the particular variable. The specific differential equation for Eq 4-15 will be derived. We have

$$F_x = \xi_x L$$

$$F_{x'} = \xi x'/L$$

$$\begin{aligned} d/dt(F_{x'}) = {} & \xi_x x' + \xi_y y' \\ & + \xi x''(L-(x'^2/L))/L^2 + \xi_t x'/L \\ & - \xi y''(x'y'/L)/L^2 \end{aligned}$$

Equation 4-15 can eventually be reduced to

$$\begin{aligned} & (\xi_x x' + \xi_y y')L^3 \\ & \quad +\xi x''(L^2 - x'^2) - \xi y''x'y' + \xi_t x'L^2 - \xi_x L^4 = 0 \end{aligned}$$

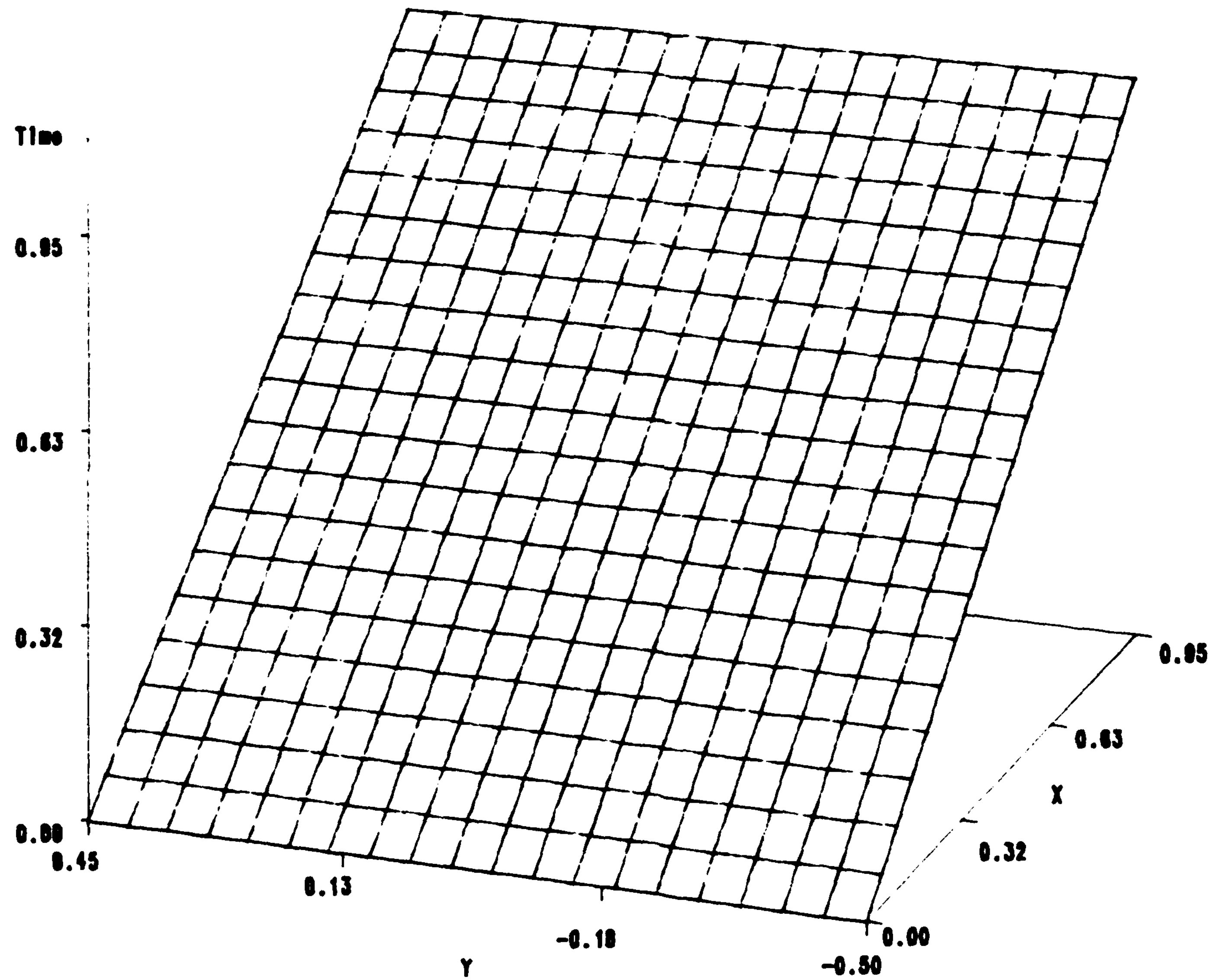
The specific Euler equation corresponding to (Eq 4-16) is

$$\begin{aligned} & (\xi_x x' + \xi_y y')L^3 \\ & \quad -\xi x''x'y' + \xi y''(L^2 - y'^2) + \xi_t y'L^2 - \xi_y L^4 = 0 \end{aligned}$$

Due to the convergence problems that arose in studying the time independent case, it was hypothesized that this

system of equations was also likely to have problems converging, remembering that ℓ is now a nonlinear function of not only x and y but also t. Because of the complexity of this system the problem of determining the optimal path was approached from a slightly different perspective.

When writing L as a function of x' and y' it was tacitly assumed that both x and y could be written as a function of t. But, has opposed to the time independent case, this does not imply that no backtracking will be allowed. So long as for any one value of t there was one and only one point specified in the x-y plane would L be satisfied. What was done then to assure that no backtracking would be allowed as well as to simplify the system of Euler equations referred to above was to assume that for the desired flight path x could be written as a linear function of t. In order to satisfy the boundary conditions that $x = 0$ at $t = 0$ and $x = 1$ at $t = t_f$ we must have $x(t) = t/t_f$. There are two implications for this simplification. First, the aircraft will never be allowed to backtrack with respect to both the x and y axes. As time increases the aircraft is pushing towards the point (1,0) in the x-y plane. The second implication is that we have reduced the number of Euler equations by one. The reason for this is that a constraint has been added, namely requiring the aircraft to fly along the geometrical plane $x - t/t_f = 0$ in the three dimensional x-y-t space. (See Fig 4-5



4-5 Flight Path Constraint
x = t

with $t_f = 1$.) The objective now is to find the path $y(t)$ that minimizes the integral

$$J(y) = \int_0^{t_f} \Phi(t/t_f, y, t)(1+(1/t_f)^2 + y'^2)^{1/2}\, dt$$

For a fixed t_f this is exactly the integral that was minimized in the time independent case with t now replacing x as the variable of integration. Letting $b = 1 + (1/t_f)^2$ the resulting system to solve then is

$$y_1' = y_2$$
$$y_2' = (b+y_2^2)(\Phi_y - \Phi_t\, y_2/b)/\Phi$$

with boundary conditions

$$y(0) = y(t_f) = 0$$

Up to this point the form of the function $\Phi(x,y,t)$ has not been specified. It was shown in equation 4-8 that for one spatial dimension, under certain boundary and initial conditions, the solution to the threat equation could be written as an infinite sum

$$\Phi(x,t) = \sum \Phi_n(x,t) \qquad 1 \le n \le \infty$$

where

$$\Phi_n(x,t) = \exp(n^2 a^2 t)\sin(n\pi x)$$

What was assumed for the particular problem at hand was that the threat potential function Φ could be written as a function of t multiplied by a function of x and y. That is, Φ was assumed to satisfy

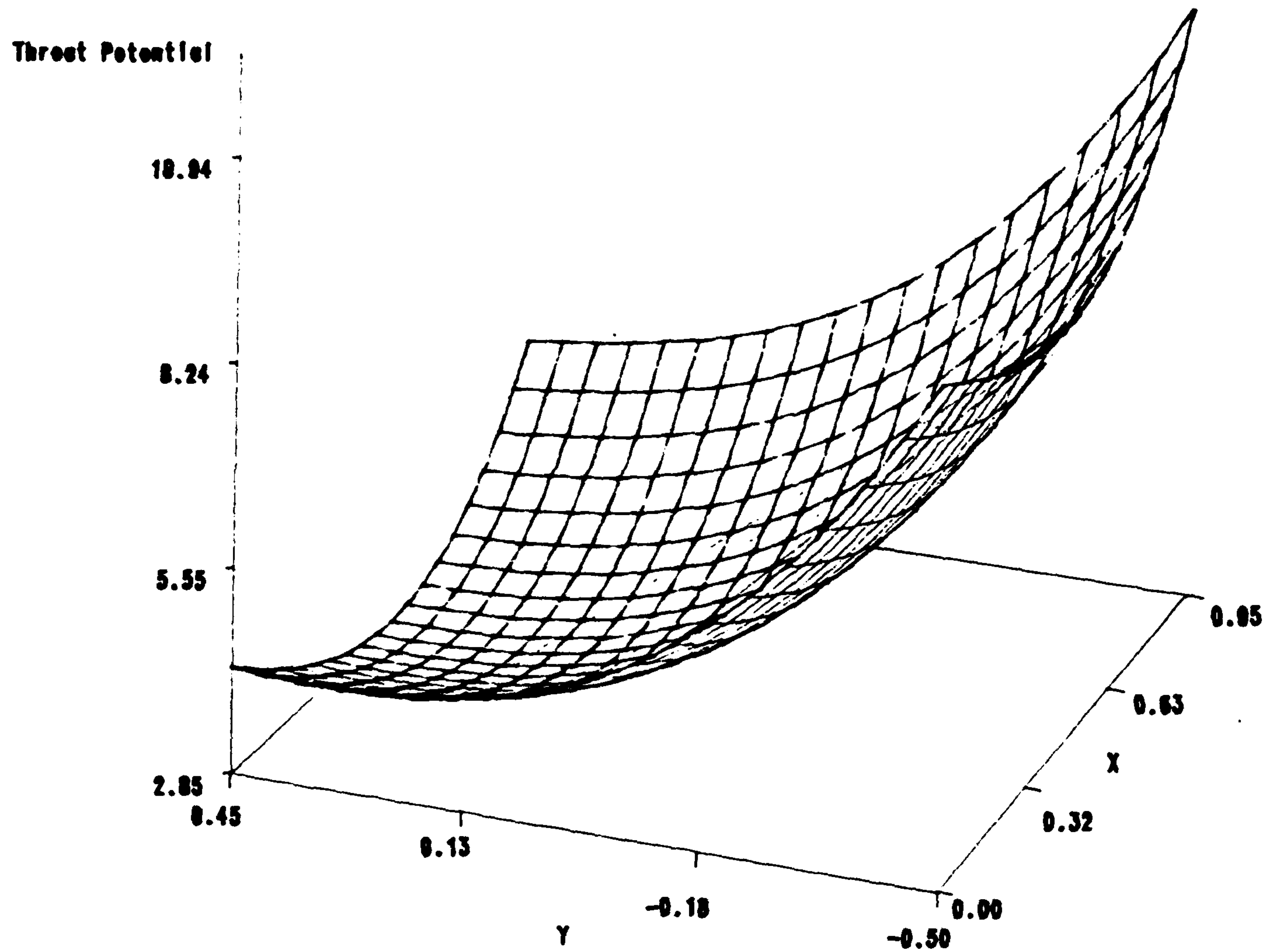
$$\Phi(x,y,t) = T(t)G(x,y) \qquad \text{(Eq 4-17)}$$

where $T(t)$ was the Taylor series expansion for $\exp(pt)$ and

$G(x,y)$ could be written as a polynomial in x and y. One interpretation of (Eq 4-17) is to consider $G(x,y)$ as a polynomial in x and y with $T(t)$ has the "amplitude" of each of the independent terms of the polynomial. One obvious drawback of (Eq 4-17) is that the coefficients of the polynomial $G(x,y)$ all change at the same rate with respect to time. This might not always be true. The main reason for assuming the threat potential could be written as (Eq 4-17) was that the system of differential equations to be solved could now be manipulated in the same way that the system was manipulated for the time independent analysis.

The threat equation was now solved. As in the time independent case y was permitted to range between -0.5 and 0.5 while both x and t ranged between 0.0 and 1.0. The reason for not allowing y to range between 0 and 1 is that the routine DVCPR requires the boundary points of the ordinary differential equation used to solve for the optimal flight path be zero. This difference in ranges between the spatial variables has no effect on the solution to the threat equation. In fact, this transformation is nothing more than offsetting the air defense zone by 0.5 distance units from the x axis.

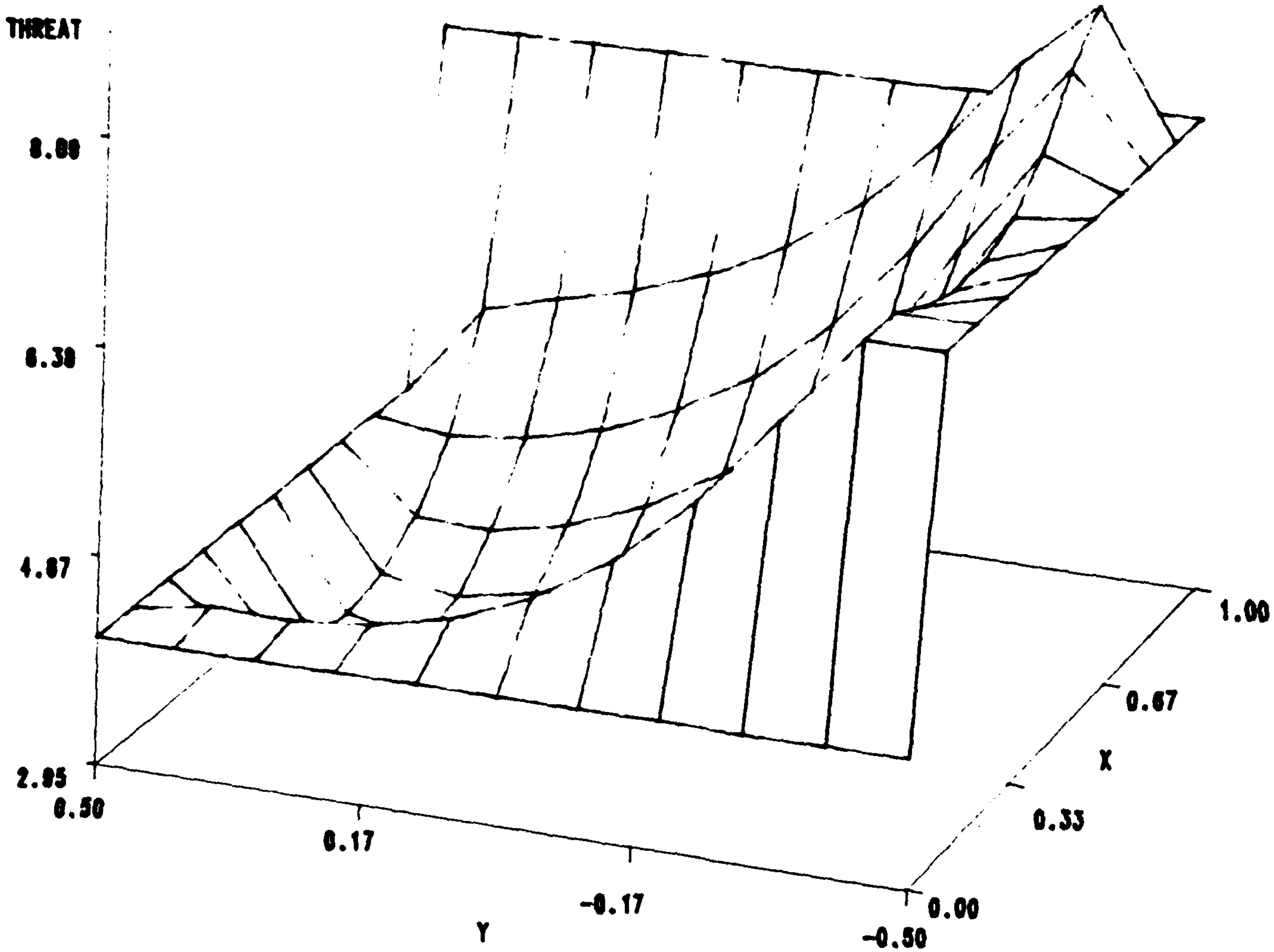
The initial conditions needed to solve the threat equation are shown in figure 4-6. The mathematical form of the initial conditions is identical to equation 4-11 and the coefficients of the independent terms are also listed in the



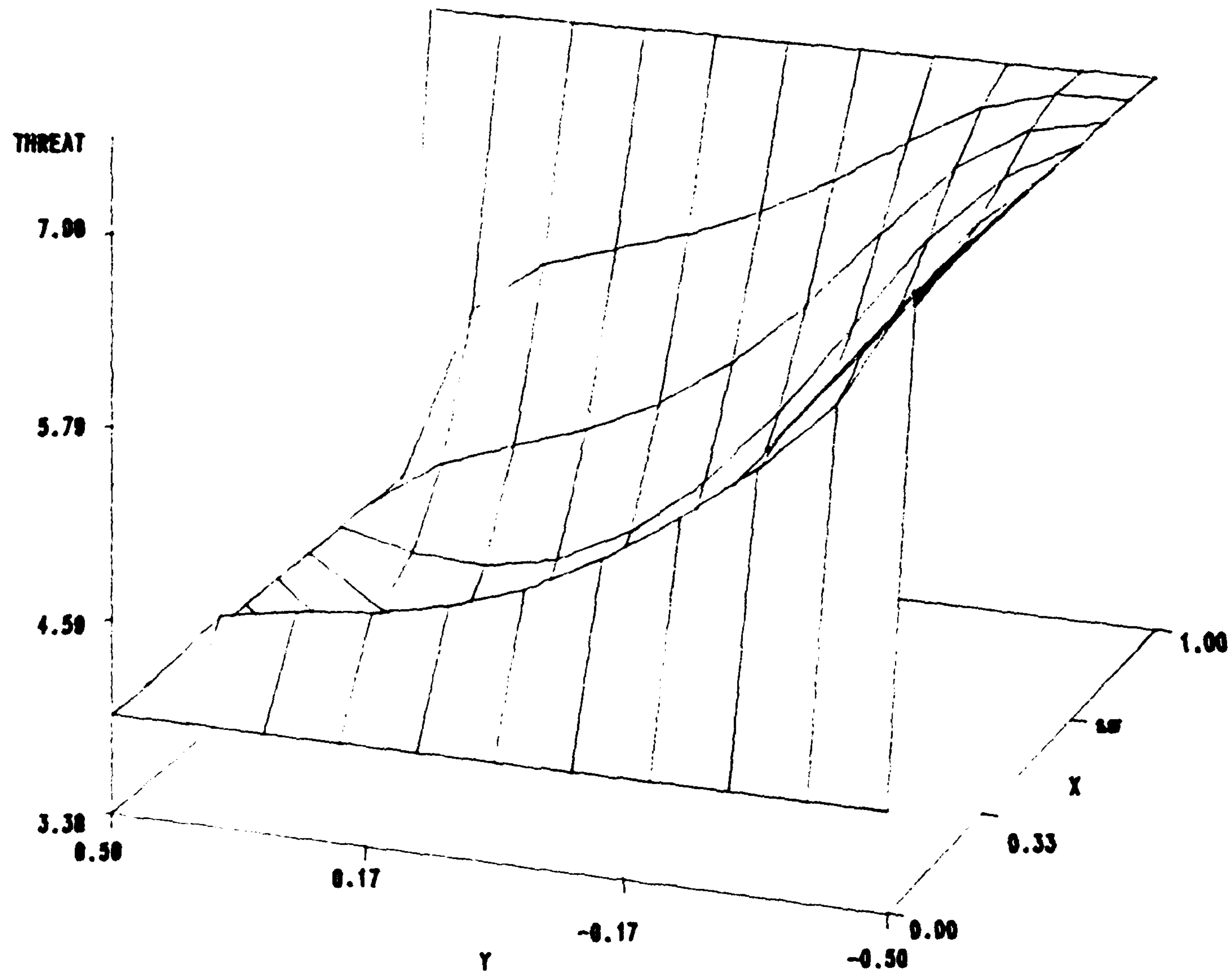
4-6 Threat Potential Initial Conditions

appendix. As was stated previously boundary conditions for all t are also required to solve a parabolic PDE. For the scenario considered, the threat potential along the y axis and along the boundary y = 0.5 were kept at 4.0 degrees (units of threat potential) while along the other two boundaries the threat potential remained 7.0 degrees. The boundary conditions are not shown in figure 4-6. The justification behind choosing these boundary conditions was that this was the minimum threat potential along the specific boundaries at t = 0 and it was assumed that the threat potential would never fall below these values. A value of 0.1 was given to the variable a^2. Finally, the threat equation was solved every 0.1 distance units and 0.05 time units between the ranges specified above. The numerical routine used to solve the threat equation was adapted from the text Numerical Analysis by Johnson and Reiss (Johnson and Reiss, 1982:466-467).

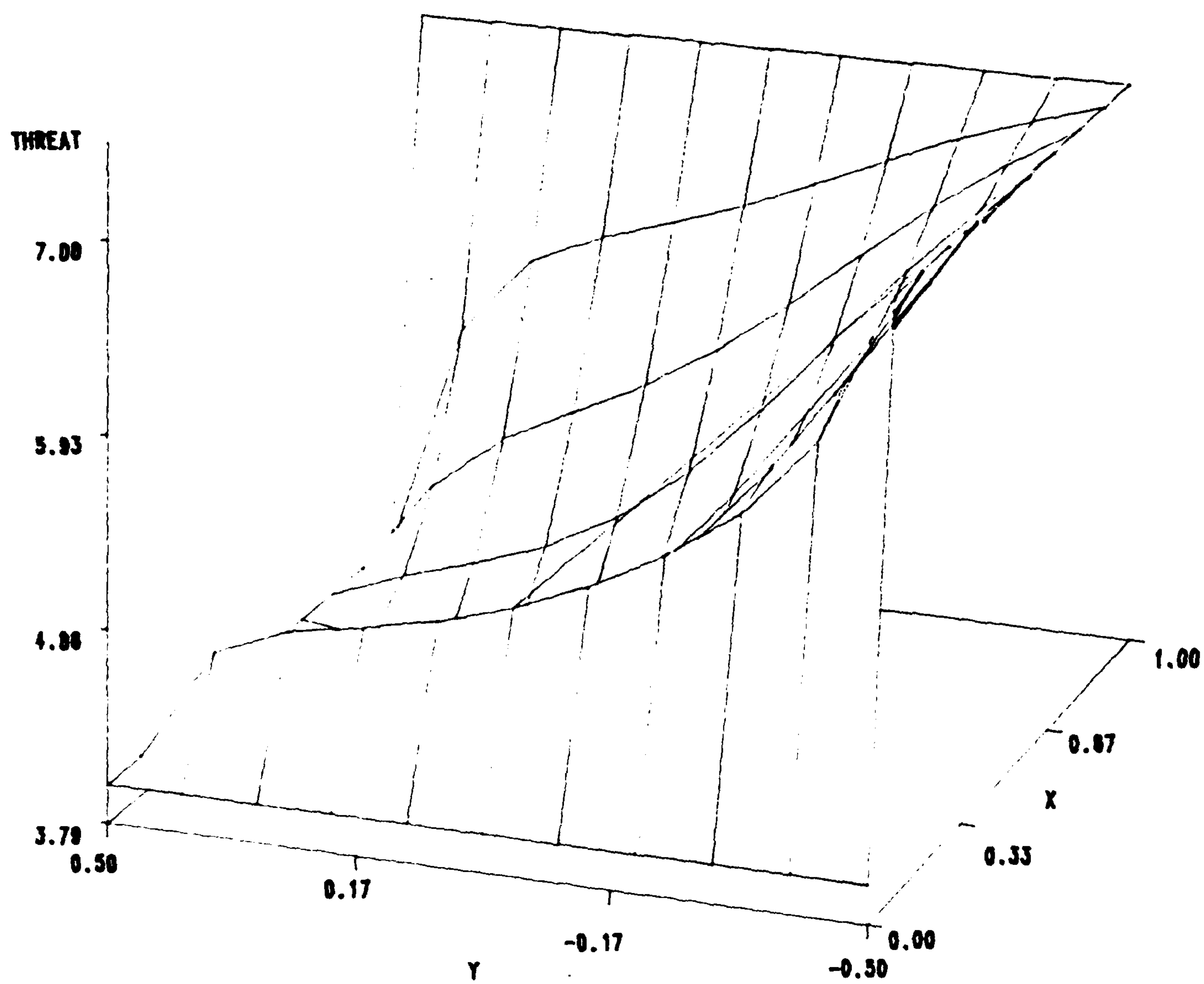
The result of solving the threat equation under the conditions stated above was that the x-y plane was divided into an 11x11 grid with the threat potential specified at each point for every 0.05 time units for a total of 2420 data points. Different threat potential contours are shown in figures 4-7 through 4-9 with the times (and now boundary conditions) shown. (Notice the decreasing scale on the threat potential axis.) It is imperative to note that the boundary conditions and the value of a^2 played as important a role in influencing the solution as did the initial conditions.



4-7 Threat Potential: t = .10



4-8 Threat Potential: t = .50



4-9 Threat Potential: t =1.0

The next step in the algorithm was to fit an equation to this data. The form of the equation hypothesized was already stated as

$$\xi(t,x,y) = T(t)G(x,y) \qquad \text{(Eq 4-17)}$$

with

$$T(t) = b1t + b2t^2 + b3t^3 + b4t^4$$

and $G(x,y)$ was again assumed to be in the same form as equation 4-11. Again the problems of nonlinear programming became apparent. The SAS system was used to determine the coefficients but after executing the program for over two hours on an almost dedicated Vax 11-780 it was determined that the system would not converge. In order to obtain coefficients what was then done was to fit $G(x,y)$ to the data at discrete intervals of time. Once the coefficients of G were known at each time t an average value of the coefficients would be used and the data generated by the threat equation would be fit to

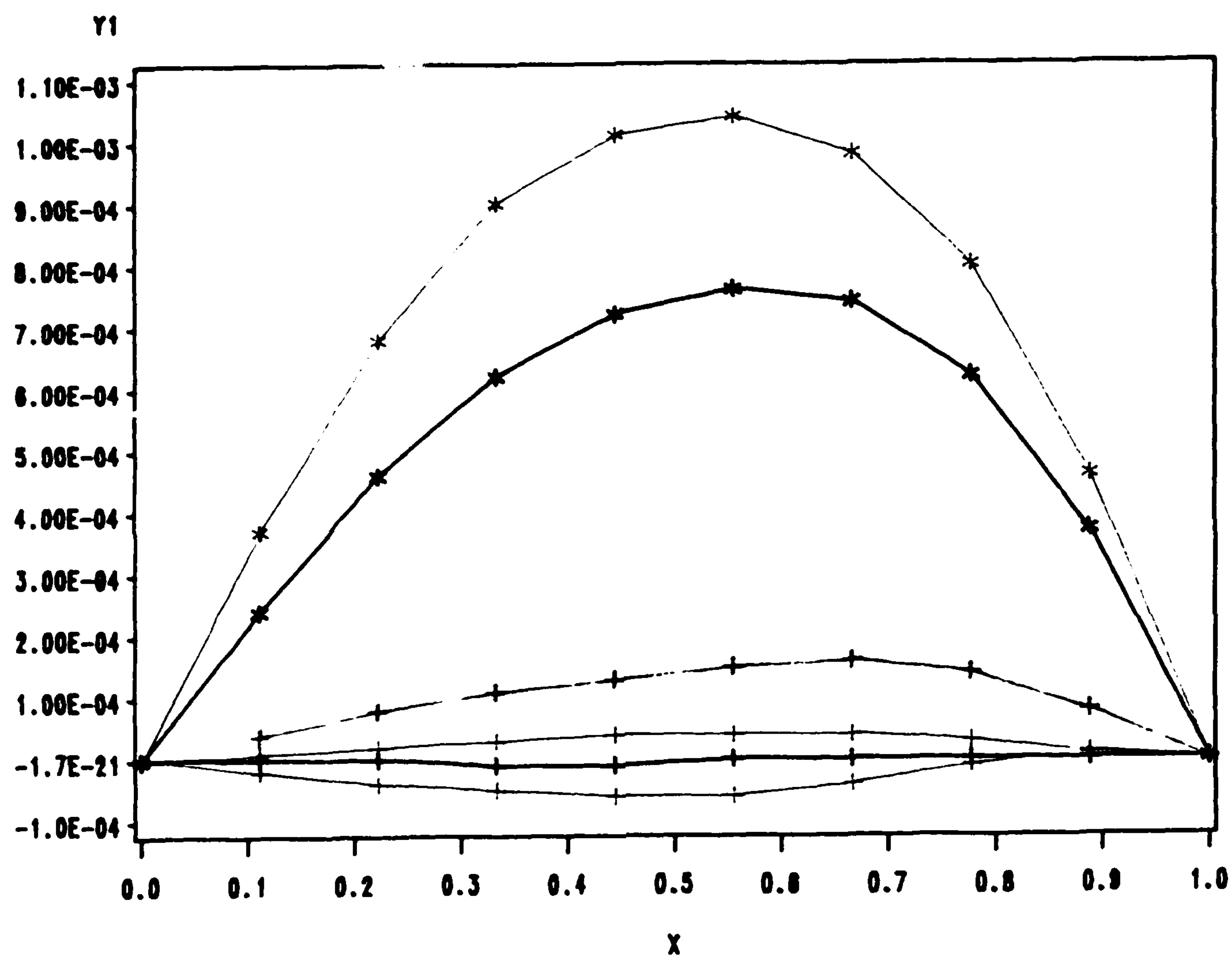
$$\xi(t,x,y) = T(t)G1(x,y) \qquad \text{(Eq 4-18)}$$

where $T(t)$ is given as above and G1 is given as in equation 4-11 but with the average value of the coefficients specified. Values of the coefficients of G at different times t are listed in the appendix. For an initial study such as this choosing the average value of the coefficients is acceptable but would not be for actual implementation. The reason for this is that, referring to the appendix, the coefficients are changing with time but not at all the same rate so that using

the average value of the coefficients is just a first approximation.

Once $\delta(t,x,y)$ was completely determined, the optimal path could be determined for different values of t_f using DVCPR. Since the system that was being solved was very similar to the time independent threat case convergence problems were expected. DVCPR has an option however for continuation of highly nonlinear problems. What continuation assumes is that the system to be solved has been embedded in a one parameter family $y' = f(t,y,epsnu)$. For epsnu = 0 the problem is simple and for epsnu = 1 the original problem is recovered. To test the program with the continuation option active, a constant threat function was assumed so that in effect the problem of finding the optimal path in this test program reduced to finding the shortest path between the points (0,0) and $(t_f,0)$ in the t-y plane. This is of course a straight line and that is what DVCPR predicted for times ranging from 0 to 3.0

It must be remembered that x(t) had already been constrained to be linear with t so that the objective was to determine y(t), or using $x(t) = t/t_f$, $y(t/t_f)$. The optimal flight paths for the different final times are shown in figure 4-10. Because of the nature of the threat potentials indicated in figures 4-6 through 4-9 it was expected that the optimal flight paths would result in flight paths that were relatively close to the x axis. This indeed was the case but



4-10 Optimal Paths: Final times of
1.5, 1.25, 1.0, 0.50, 0.40, 0.75

there were two unexpected results. First, the optimal flight path for $t_f = 0.75$ was below the y axis while all other flight paths were above. Second, although the flight paths differed significantly between themselves, relative to the size of the air defense zone there was almost no difference. Attention should be drawn to the scale of the y axis used in figure 4-10 which is significantly smaller than the scale of the y axis used for the air defense zone.

One final comment, attempts were made to determine the optimal flight paths for times greater than 1.5 but in all cases convergence was not achieved.

<u>Summary</u>

An algorithm for obtaining an optimal flight path was presented which utilized the calculus of variations. The threat equation was solved in both one and two spatial dimensions. For the one dimensional case the dependence of the threat potential on aircraft speed was explicitly shown. Optimal paths were obtained for a time varying threat potential as well as for a scenario where the threat potential was time independent.

V. Implications and Recommendations

This chapter contains suggestions for investigating the behavior of the solution to the threat equation under different scenarios as well as some implications for the use of the methodology presented in this thesis. The threat equation can be approached from two perspectives. The first approach was taken in this thesis, that is, given that the threat to an aircraft operating in an air defense zone changes with respect to position, velocity, and time according to the threat equation, what is the "optimal" path through the threat? As was stated in chapter 1, if the optimal path predicted was not intuitively appealing then that would be an indication that the threat equation which was proposed was not accurately describing the real world. In effect, this thesis presented two results. The first result, presented in chapter 3, was a mixed heuristic and mathematical justification for using the threat equation. The second result, contained in chapter 4, was a preliminary attempt at using the threat equation and the calculus of variations techniques to determine flight paths.

The second approach that can be taken with respect to the threat equation is a control theory approach. The control theory approach should be investigated but it will not be discussed further due to the fact that any presentation of the theory in the recommendations chapter of a thesis would be cursory at best.

Review of Objectives

The primary objective of this research was to investigate the feasibility of describing the threat to an aircraft operating in an air defense zone as the solution to a partial differential equation. A methodology for determining the optimal flight path through the threat field was proposed as a way of partially validating the use of the particular partial differential equation presented. In order to accomplish the primary objective five subobjectives were proposed at the end of chapter one. The first three subobjectives relate to the derivation of the threat equation and were accomplished in chapter three. A system for the setting of initial and boundary conditions needed to solve a parabolic partial differential equation, and which also took into account pilot inputs', was proposed in chapter two. A fourth subobjective was to obtain computer resources so as to be able to solve not only a partial differential equation but the ordinary differential equations that arose in determining the optimal flight path. The fifth and final subobjective was to validate the predicted optimal flight paths with intuition. This final subobjective was accomplished for one time dependent threat scenario.

Strengths and Weaknesses

The optimal path approach has obvious practical implications but there are weaknesses to the methodology.

Initially the hope was that the approach taken would require less computer resources than that used by other one-on-many models. The results here were mixed. On the plus side was the fact that once the threat could be described in closed form the run time to determine the optimal path was minimal. Admittedly Euler's equations had to be simplified in order to assure convergence but the resulting flight paths for the scenario considered were not greatly different from what intuition would predict. Unfortunately, in order to get the threat into a closed form the cpu time needed (on a Vax 11-780) was quite extensive. As was stated in chapter 4 initial attempts at fitting the threat data to a function took over two hours to execute with the result that convergence never was achieved. This is the nature of nonlinear programming.

As previously stated, it is conceivable that the optimal flight path for some scenario could result in a path which is safe for most of the flight but for a brief instant of time requires flying through a large threat potential. This however could be handled in one of two ways. First, redefine the measure of effectiveness so that the optimal flight path is the flight path which minimizes the maximum value of the threat for a flight path. This would result, however, in not being able to utilize the calculus of variations as the optimization scheme. A different method for overcoming a flight path through a threat peak, and one which would still allow for a calculus of variations approach, is to alter the

initial conditions in such a way so as to prohibit the flight through the threat peak. The threat as a function of distance (from a group of threat sources) was approximated as a $1/r^2$ relationship. Technically, at zero distance from the threat source there is infinite threat. Since it is impossible to represent infinity on a computer some large number must be used. Under the assumption that the undesirable flight path would achieve the peak threat value at a position relatively close to if not at a point of threat infinity then that large number used to represent infinity should be made larger. It is not necessary, in fact it is probably not desirable, to alter the large number which is specified at all threat sources but only at the threat sources which correspond to the position of peak threat in the flight path.

The main strength of being able to describe the threat to an aircraft operating in an air defense zone as a solution to a partial differential equation is that not only can an optimal flight path be determined but the threat at any point in space-time for any flight path can also be calculated. If another flight path is thought to be optimal with respect to some other measure of effectiveness then this flight path can be compared to any other "optimal" flight path at any point in the flight path to see where the points of disparity, or large differences in threat potential, occur.

Recommendations for Further Research

Full Validation. Only one scenario has been investigated

in this thesis. The next step is to consider a full validation of the threat equation for a wide range of scenarios.

Initial Conditions. Schemes other than the $1/r^2$ relationship for expressing the threat as a function of distance from the threat source should be investigated. This inverse square relationship of threat with distance is a first approximation and was proposed due to the symmetry of this function about the point $r = 0$. Some threat sources will not have the ability to inflict damage for the full 360 degrees about the threat source.

Ratio of k with σc. The numerical value that the ratio $k/\sigma c$ assumes is a very important influence on the behavior of the solution to the threat equation. Assume two pilots, flying the same type of aircraft, performing the same mission, at the same altitude and speed, through an air defense zone in which pilot one has rated according to his scale. Assume the first pilot has rated the total source strength of the zone at 60 fp's. When pilot two rates the same air defense zone he rates each one of the threat sources at ,for example, five times what pilot one has rated them for a total source strength of 300 fp's. Technically, each of the pilots have rated the zone and all the threat sources the same but the second pilot has used a different threat source

strength scale. The result is that for the second pilot the threat equation predicts that the threat will change S times slower than it will for the first. This is unacceptable since there really is no difference between the pilot ratings and all the other variables are held constant. One obvious remedy for this situation is to allow the source strength scale to range between 0 and S where S is some upper limit.

Once the upper limit on the source strength scale is chosen then different values of k will have to be investigated to see what the best ratio of k to σc is. From chapter 4 k is proportional to wU where U is the speed of the aircraft and w is a positive scalar. The value of w should be varied with all other variables (including boundary and initial conditions) held constant while different solutions to the threat equation are analyzed. Three dimensional graphics capabilities are extremely helpful for this analysis. There is no hard and fast rule to determine the "right" k, again intuition must be used. However this is not a tremendous shortcoming for many scenarios. Initially the threat equation was solved with $a^2 = 1$. Once the solution was generated and plotted on a 3-D graphics terminal it was obvious that the threat potential was decreasing too fast to be realistic.

<u>Threat Generation Function</u>. Different threat generation functions should be investigated and their effect on the

solution of the threat equation noted. This, by and large, is quite simple. What is not simple is assigning these mathematical functions some real world meaning. As was stated the threat generation function is the sum of a threat source function and a threat sink function, therefore both of these functions is what is really being investigated and given the real world meaning.

The threat source function is probably the easier of the two functions to investigate due to the fact that the location of a source of threat is, ideally, known. However a threat source function which is varying with time is not only harder to solve mathematically but is also harder to assign some real world meaning. This is the case when considering a time varying threat source function at a single threat source. It would possibly be easier to consider a threat source function for the entire air defense zone (ie- a function of time alone). This is usually easier to solve and for certain functions easier to give some meaning to. As an example a threat source function of $t^n \exp(-t)$ could be interpreted as the threat increasing rapidly for the first few time units while the threat sources are becoming operational and then eventually leveling off to a constant while the battle takes place.

The threat sink function is more difficult to investigate because to decrease the threat in an air defense zone the aircraft or some supporting forces must perform

offensive operations against the threat sources. Again, it might be easier to consider a threat sink function acting uniformly throughout the entire zone instead of modeling offensive operations against one of the threat sources.

Although it was stated at the very beginning of this thesis that the use of the probability of kill as a measure of effectiveness has some drawbacks it was not meant to imply that the scenarios considered are entirely deterministic. There is always the chance that the threat might not be exactly what the threat equation predicts. A stochastic term could be introduced into the threat equation. For that case the differential equation would become a stochastic differential equation.

<u>Closed Form Solutions.</u> As a final recommendation, different equations should be investigated in the course of trying to fit the data generated by the threat equation to a closed form solution. The greatest shortcoming in the implementation of this methodology is the great need for computer resources for the fitting of the data to an equation. As an initial attempt fitting the data to a sine expansion instead of a polynomial in x and y should be investigated. The overriding goal, as with any statistical investigation, should be to use the least amount of computer resources while still maintaining the features of the threat data.

Summary

It appears that describing the threat to an aircraft operating in an air defense zone as the solution to a partial differential equation is feasible and therefore further research should be conducted. One partial differential equation has been proposed in this thesis with a validation scheme. Full validation of the threat equation is needed.

Appendix: Polynomial Coefficients

Below is the table of the coefficients for each of the polynomials used in this thesis. The coefficient A20 corresponds to x^2, A11 to xy, A12 to xy^2, etc. Column 1 is the particular scenario under consideration with the legend following the table.

S	A30	A21	A12	A03	A20	A11	A02	A10	A01	A0C
1	-8.44	00.00	-13.16	7.89	8.02	0.00	2.59	0.0	-1.15	0.55
2	5.00	*****	*****	-5.0	1.00	-1.00	8.50	-4.0	-3.75	4.68
3	12.37	14.09	-2.58	7.55	-13.75	-14.36	5.86	4.49	-2.31	3.70
4	12.39	14.14	-2.66	6.84	-13.91	-14.37	5.65	4.58	-2.08	3.72
5	12.41	14.12	-2.73	6.16	-14.07	-14.32	5.46	4.71	-1.88	3.74
6	12.43	14.05	-2.77	5.52	-14.21	-14.22	5.29	4.84	-1.70	3.76
7	12.44	13.95	-2.81	4.91	-14.38	-14.10	5.13	4.96	-1.55	3.77
8	12.45	13.71	-2.86	3.81	-14.58	-13.82	4.84	5.19	-1.30	3.81
9	12.45	13.45	-2.90	2.85	-14.79	-13.54	4.58	5.40	-1.09	3.84
10	12.46	13.19	-2.92	2.01	-15.00	-13.26	4.33	5.60	-0.92	3.86
11	12.44	12.25	-2.97	-.63	-15.79	-12.28	3.43	6.41	-0.42	3.96
12	12.43	13.66	-2.80	4.33	-14.49	-13.81	4.95	5.13	-1.47	3.80

Scenario 1= Time independant case

Scenario 2= Time dependant case; initial conditions

****: An extra term was added for the initial conditions. The term was $(xy)^2$. The coefficient was A22 = -1.0

Scenario 3-11= Time dependant cases for

t = 0.05, 0.10, 0.15, 0.20, 0.25, 0.35, 0.45, 0.55, 1.00

Scenario 12= Average value of coefficients for scenarios

t = 0.05 through t = 1.00

The coefficients for $I(t)$ in the time dependent threat case was

b1= 11.40
b2=-38.58
b3= 50.11
b4=-22.01

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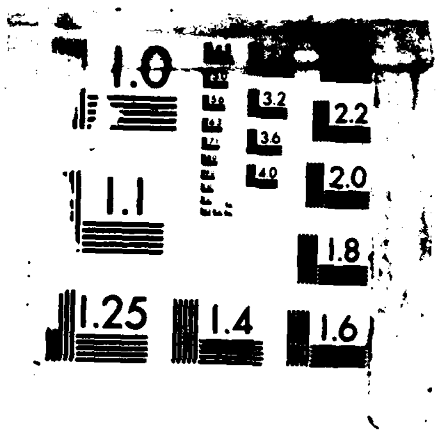
1.0
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

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ABSTRACT

This research investigates the feasibility of describing the threat potential at a point in a specified air defense zone as a function of position and time while stipulating that the function is a solution to a partialdifferential equation. The mission, the aircraft, and the hostility of the air defense zone are incorporated into the forcing function, as well as the initial and boundary conditions which are needed to solve the partial differential equation. A constant speed and altitude of the aircraft is assumed.

In order to validate the partial differential equation used to generate the threat data a scheme for determining the path of least threat through the threat data is proposed. The calculus of variations, a branch of mathematics concerned with the optimization of an integral, is used to find the curve x(t) and y(t) which minimizes the total threat of the flight path. Preliminary results indicate that it does appear feasible to describe the threat as above but furthervalidation is required.

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